Precise definition of $\to$ and $\to_{fin}$: Let $A = (Q, q_0, \to, Q_F)$ an NBA.

We have $q \xrightarrow{\varepsilon} q$ for all $q \in Q$ and $q_f \xrightarrow{\varepsilon}_{fin} q_f$ for all $q_f \in Q_f$.

For all $q, q' \in Q$ and $u = a_1 \dots a_n \in \Sigma^+$, we set

$q \xrightarrow{u} q'$ if there are $q_1, \dots, q_{n-1} \in Q$ so that

$$q = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n = q'.$$

Note that $n = 1$ is possible so that there are no intermediary $q_1, \dots, q_{n-1}$.

We furthermore have

$q \xrightarrow[fin]{u} q'$ if the $q_1, \dots, q_{n-1} \in Q$ can be chosen so that

$$q_i \in Q_F \text{ for some } 0 \le i \le n,$$

(including $q = q_0$ and $q_n = q'$).

Recapitulation:

Goal: Solve universality $\mathcal{L}(A) = \Sigma^\omega$.

Approach: Investigate classes $\sim_A \subseteq \Sigma^* \times \Sigma^*$ induced by A.

Remember:

$u \sim_A v$ if for all $q, q' \in Q$: $q \xrightarrow{u} q'$ iff $q \xrightarrow{v} q'$
and
$q \xrightarrow{u}_{fin} q'$ iff $q \xrightarrow{v}_{fin} q'$.

Observation:

Characterise equivalence class $[u]_{\sim_A}$ by two relations on states:

$R_{[u]_{\sim_A}} \subseteq Q \times Q$ defined by $R_{[u]_{\sim_A}} = \{(q, q') \mid q \xrightarrow{u} q'\}$

$R_{[u]_{\sim_A}} \subseteq Q \times Q$ defined by $R^{fin}_{[u]_{\sim_A}} = \{(q, q') \mid q \xrightarrow{u}_{fin} q'\}$.

How to find the equivalence classes (their state-based representations):

$\hookrightarrow$ Start with $R_{[a]_{\sim_A}}$ and $R^{fin}_{[a]_{\sim_A}}$ for $a \in \Sigma$.

↪ Derive state-based representation for further classes $[uv]_{\sim_A}$ by composition:

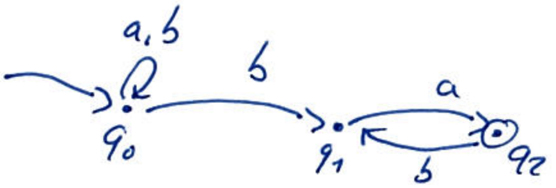
$$R_{[u.v]_{\sim_A}} = R_{[u]_{\sim_A}} ; R_{[v]_{\sim_A}}$$

$$R^{fin}_{[uv]_{\sim_A}} = (R^{fin}_{[u]_{\sim_A}} ; R_{[v]_{\sim_A}}) \cup (R_{[u]_{\sim_A}} ; R^{fin}_{[v]_{\sim_A}}).$$

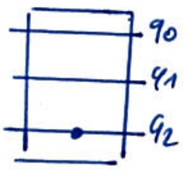
⇒ Until a fixed point is reached, i.e., no more classes are found.
(such a fixed point is guaranteed to be found as there are only finitely many classes).

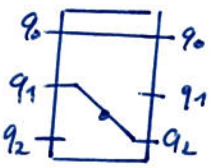
Example:

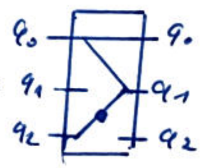
A



Represent $[u]_{\sim_A}$ by circuits that summarise $R_{[u]_{\sim_A}}$ and $R^{fin}_{[u]_{\sim_A}}$:

$[\varepsilon]_{\sim_A} =$ 

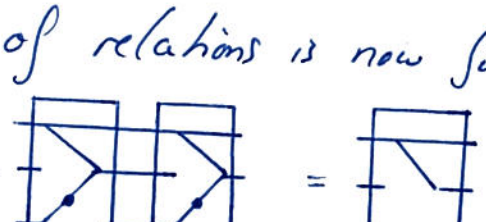
$[a]_{\sim_A} =$ 

$[b]_{\sim_A} =$ 

In circuit $[u]_{\sim_A}$ • we have q — q' if $q \xrightarrow{u} q'$

• we have q —•— q' if $q \xrightarrow{u}_{fin} q'$.

Composition of relations is now found by connecting circuits:

$[bb]_{\sim_A} =$ 

Enumerating (at most) all possible classes yields finite multiplication table.

The following theorem is a consequence of Büchi's complementation procedure (negate the statement, compare to $\overline{L(A)} \neq \emptyset$).

<u>Theorem (Fogarty & Vardi '10):</u>

Consider an NBA A.

We have $\mathcal{L}(A) = \Sigma^\omega$ iff

for all $[u]_{\sim A}, [v]_{\sim A}$ with $[u.v]_{\sim A} = [u]_{\sim A}$

and $[v.v]_{\sim A} = [v]_{\sim A}$ with $[v]_{\sim A} \cap \Sigma^+ \neq \emptyset$ we have

there is $q \in Q$ with $(q_0, q) \in R_{[u]_{\sim A}}$ and $(q,q) \in R^{fin}_{[v]_{\sim A}}$.

<u>Proof:</u>

"$\Rightarrow$" Let $A = (Q, q_0, \to, Q_F)$ with $\mathcal{L}(A) = \Sigma^\omega$.

Consider classes $[u]_{\sim A}$ and $[v]_{\sim A}$ with $[u.v]_{\sim A} = [u]_{\sim A}$

and $[v.v]_{\sim A} = [v]_{\sim A}$.

We have to find $q \in Q$ as required.

Assume wlog. $v \neq \varepsilon$, and consider an accepting run of A on $u.v^\omega$ (accepting run exists by universality).

By pigeon hole principle, some state $q \in Q$ is visited infinitely often in

$$q_0 \xrightarrow{u.v^{i_0}} q \xrightarrow{v^{i_1'}} q \xrightarrow{v^{i_2'}} q \xrightarrow{v^{i_3'}} \ldots \quad \text{for } i_0, i_1', i_2', i_3', \ldots > 0.$$

By acceptance, there are infinitely many final states in between. So we can assume

$$q_0 \xrightarrow{u.v^{i_0}} q \xrightarrow{v^{i_1}}_{fin} q \xrightarrow{v^{i_2}}_{fin} q \xrightarrow{v^{i_3}}_{fin} \ldots \quad \text{for } i_0, i_1, i_2, i_3, \ldots > 0.$$

Since $[u.v]_{\sim A} = [u]_{\sim A}$, we have $[u.v^{i_0}]_{\sim A} = [u]_{\sim A}$. So $(q_0, q) \in R_{[u]_{\sim A}}$.

Similarly, by $[v.v]_{\sim A} = [v]_{\sim A}$, we have $[v^{i_j}]_{\sim A} = [v]_{\sim A}$ for all $j > 0$.

So $(q,q) \in R^{fin}_{[v]_{\sim A}}$ as required.

"$\Leftarrow$" Assume all classes $[u]_{\sim_A}, [v]_{\sim_A}$ with $[uv]_{\sim_A} = [u]_{\sim_A}$ and $[v.v]_{\sim_A} = [v]_{\sim_A}$ have a state $q \in Q$ as required.

We have to show universality, $L(A) = \Sigma^\omega$.

Let $w \in \Sigma^\omega$, we show that $w \in L(A)$.

To this end, give an accepting run.

<u>Apply Ramsey's theorem:</u>

Color $(\mathbb{N}, E)$ with the $\sim_A$ equivalence classes:

$$f(\{i,j\}) = [a_i \dots a_{j-1}]_{\sim_A} \quad \text{for } i<j.$$

Then there is an infinite subset $S \subseteq \mathbb{N}$ with

$$f(\{i,j\}) = [v]_{\sim_A} \quad \text{f.a. } i<j \text{ in } S$$

Let i_j belong to S:

Then

$$\in S$$

$$w = a_0 \dots \underbrace{\underbrace{a_{i_0} \dots a_{i_1-1}}_{\sim_A v} \underbrace{a_{i_1} \dots a_{i_2-1}}_{\sim_A v}}_{\sim_A v} \underbrace{a_{i_2} \dots a_{i_3-1}}_{\sim_A v} a_{i_3} \dots$$

Pic $u := a_0 \dots a_{i_1} - 1$.

Then
$$\begin{aligned}
[u.v]_{\sim_A} &= [a_0 \dots a_{i_0} \dots a_{i_1-1} \cdot v]_{\sim_A} \\
&= [a_0 \dots \underbrace{a_{i_0} \dots a_{i_1-1} \cdot \overbrace{a_{i_1} \dots a_{i_2-1}}}]_{\sim_A} \\
&= [a_0 \dots a_{i_0-1} \cdot v]_{\sim_A} \\
&= [a_{i_0} \dots a_{i_0-1} \cdot \overbrace{a_{i_0} \dots a_{i_1-1}}]_{\sim_A} = [u]_{\sim_A}.
\end{aligned}$$

For $[vv]_{\sim_A} = [v]_{\sim_A}$, the argumentation is similar.

By the assumption on the equivalence classes, there is $q \in Q$ with

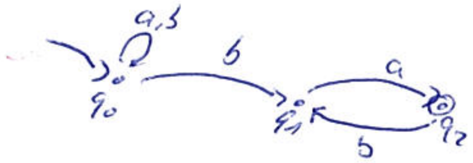
$$(q_0, q) \in R_{[u]_{\sim_A}} \quad \text{and} \quad (q,q) \in R^{fi}_{[v]_{\sim_A}}.$$

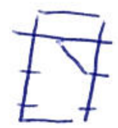
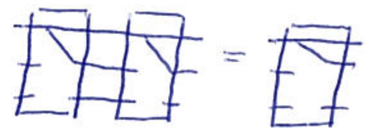
This yields an accepting run of A on w. $\square$

Algorithmically:

- Try every idempotent equivalence class $[vv]_{\sim_A} = [v]_{\sim_A}$
  ↳ is there $[u]_{\sim_A}$ with $[uv]_{\sim_A} = [u]_{\sim_A}$
  so that for all $q \in Q$:
  $$q_0 \xrightarrow{u} q \text{ implies } q \overset{v}{\not\to}_{fin} q \;?$$
  Then A does not accept all words.

In the example:



We have $[bb]_{\sim_A} =$  with $[bb.bb]_{\sim_A} =$ 

Pic $[u]_{\sim_A} = [v]_{\sim_A} = [bb]_{\sim_A}$

This means
$$[u.v]_{\sim_A} = [u]_{\sim_A} \text{ and } [vv]_{\sim_A} = [v]_{\sim_A}.$$

We have
$$q_0 \xrightarrow{bb} q_0 \quad \text{but} \quad q_0 \overset{bb}{\not\to}_{fin} q_0$$
and
$$q_0 \xrightarrow{bb} q_1 \quad \text{but} \quad q_1 \overset{bb}{\not\to}_{fin} q_1.$$

Indeed: $bb.(bb)^\omega \notin L(A)$.

Remark:

↳ Stop construction when first counterexample has been found.
↳ If $L(A) = \Sigma^\omega$, construct full table. ⟹ How to find counterexamples systematically?
⟹ How to find $[u.v]_{\sim_A} = [u]_{\sim_A}$?
⟹ How to stop algorithm earlier in case of success?
↳ This algorithm seems to work well in parallel.
(⟹ Bachelor's or Master's theses).
(Check Concur 2011, list of accepted papers, Abdulla).