Recapitulation:

Goal: $\mathcal{A} \models \varphi$ with φ in LTL

Definition of LTL:

- Finite set of propositions $(p, q, \ldots \in) \, P$
- Letters are sets of propositions, $\Sigma = \mathbb{P}(P)$ $(\ni a)$
- Formulas evaluated in positions of words w in Σ^ω:

  $w, i \models \varphi$.

Syntax of LTL:

$$\varphi ::= p \mid \varphi \vee \psi \mid \neg \varphi \mid \underbrace{O\varphi}_{\text{"next } \varphi\text{"}} \mid \underbrace{\varphi \,\mathcal{U}\, \psi}_{\text{"}\varphi \text{ until } \psi\text{"}}$$

Abbreviations:

$\underbrace{\Diamond \varphi}_{\text{"eventually } \varphi\text{"}} := \text{true} \,\mathcal{U}\, \varphi \qquad \underbrace{\Box \varphi}_{\text{"globally/always } \varphi\text{"}} := \neg \Diamond \neg \varphi$

$\underbrace{\varphi \,\mathcal{R}\, \psi}_{\text{"}\varphi \text{ releases } \psi\text{"}} := \neg(\neg \varphi \,\mathcal{U}\, \neg \psi)$

Size of a formula:

$|p| := 1 \qquad |\neg \varphi| := 1 + |\varphi| =: |O\varphi|$

$|\varphi \mathbin{\substack{\vee\\ \wedge\\ \mathcal{U}\\ \mathcal{R}}} \psi| := |\varphi| + 1 + |\psi|$.

Logical equivalence $\equiv$:

$\varphi \equiv \psi$ if for all $w \in \Sigma^\omega$ and all $i \in \mathbb{N}$

$w, i \models \varphi$ iff $w, i \models \psi$.

Some language-theoretic considerations:

Every letter $a \in \Sigma$ can be described precisely by characteristic formula:

$$\chi_a := \bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \neg p.$$

With this, capture languages over Σ by LTL formulas.

Language $(a.b)^\omega$ defined by

$$\chi_a \wedge \Box((\chi_a \to \bigcirc \chi_b) \wedge (\chi_b \to \bigcirc \chi_a)).$$

Language $\underbrace{(a.(a \cup b))^\omega}_{\text{"even positions have an a"}}$ not LTL-definable.

<u>Why?</u>

↳ LTL-definable languages definable in FO on infinite words

↳ Recall:

↳ Words of even length not definable in FO on finite words

↳ Similar argument here.

<u>Positive normal form and properties of until</u>

<u>Definition (Positive normal form)</u>:

Let $\mathcal{P}$ a finite set of propositions.

The LTL formula over $\Sigma = \mathbb{P}(\mathcal{P})$ is in <u>positive normal form</u> if it is constructed from

$p, \neg p$ with $p \in \mathcal{P}$ and $\vee, \wedge, \bigcirc, \mathcal{U}, \mathcal{R}$.

<u>Lemma:</u>

Every LTL formula φ over Σ is logically equivalent to a formula ψ in positive normal form with $|\psi| \leq 2|\varphi|$.

<u>Proof</u>:

Use following equivalences:

- $\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$
- $\neg(\varphi \,\mathcal{U}\, \psi) \equiv \neg(\neg(\neg\varphi) \,\mathcal{U}\, \neg(\neg\psi)) \equiv \neg\varphi \,\mathcal{R}\, \neg\psi$
- $\neg(\varphi \,\mathcal{R}\, \psi) \equiv \neg\varphi \,\mathcal{U}\, \neg\psi.$

$\Box$

For translation of LTL into Büchi automata,
use "<u>unrolling</u>" of $\mathcal{U}$ (also called <u>inductive property</u>)

<u>Lemma</u>:
For all $\varphi, \psi \in LTL$ we have $\varphi \mathcal{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi))$.

<u>Proof</u>: Homework.

Logical equivalence $\equiv$ in LTL is in fact a <u>congruence</u>:
If $\varphi \equiv \psi$ and φ is part of a larger formula $\Theta(\varphi)$,
then $\Theta(\varphi) \equiv \Theta(\psi)$.

As a consequence:

$$\begin{aligned}\varphi \mathcal{U} \psi &\equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi)) \\ &\equiv \psi \vee (\varphi \wedge \bigcirc(\psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi)))) \\ &\equiv \ldots\end{aligned}$$

Gives a means to check $\varphi \mathcal{U} \psi$ at position i:
- either ψ holds
- or φ holds and $\varphi \mathcal{U} \psi$ holds in next position $i+1$.

Have to ensure ψ eventually holds (unrolling happens only finitely many times).
↳ Find states forbid infinite unrollings.
⇒ Following procedure exploits unrolling.

6.2 From LTL to NBA

<u>Goal</u>: Translate LTL into NBA
↳ without using intermediary FO representation
↳ and then Büchi's result.
Why is LTL easier?
↳ Like automaton only looks into future
↳ Do not follow inductive structure of formulas
⇒ safes determinisation / complementation

at each negation
⇒ and thus exponential blow-ups.
↳ Instead, keep track of satisfaction of all subformulas while reading input.

Definition (Generalised NBA):
A generalised nondeterministic Büchi automaton GNBA is a tuple
$$A = (Q, Q_I, \rightarrow, (Q_F^i)_{1 \le i \le k})$$
with
- set of initial states $Q_I \subseteq Q$ (instead of $q_0 \in Q$)
- family of final states $(Q_F^i)_{1 \le i \le k}$

A run is still
$$r = q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} \dots \quad \text{with } q_0 \in Q_I.$$
A run is accepting if
$$Inf(r) \cap Q_F^i \neq \emptyset \quad \text{f.a. } 1 \le i \le k$$
"Every set of final states is visited infinitely often"

"Generalisation" does not increase expressiveness of the automaton model.

Lemma:
For every GNBA $A = (Q, Q_I, \rightarrow, (Q_F^i)_{1 \le i \le k})$
there is an NBA $A' = (Q', q_0, \rightarrow', Q_F')$
with $L(A) = L(A')$ and $|Q'| = |Q|k + 1$.

Idea:
Use counters from intersection construction:
$$L(A) = \bigcap_{1 \le i \le k} L(A_i) \quad \text{with } A_i = (Q, Q_I, \rightarrow, Q_F^i)$$

Construction (directly):

↳ Several initial states to one ⇝ pic new state

↳ Several sets of final states to one:

⇒ use counters in new states: $Q' = Q \times \{1, \ldots, k\}$

⇒ (q, i) means: next final state is expected from Q_F^i.

⇒ new final states: $Q_F^0 \times \{0\}$

(this choice is arbitrary, could be any $1 \leq i \leq k$). □

Idea of the translation:

↳ Subformulas of $\theta \in LTL$ as states in the automaton

↳ Intuition: formulas that currently hold.

Definition (Fischer-Ladner closure):

Let θ an LTL formula in positive normal form.

Its Fischer-Ladner closure $FL(\theta)$ is the smallest set of LTL formulas in positive normal form so that

- $\theta \in FL(\theta)$ and
- If $\varphi \overset{\vee}{\wedge} \psi \in FL(\theta)$ then $\{\varphi, \psi\} \subseteq FL(\theta)$.
- If $\varphi \mathcal{U} \psi \in FL(\theta)$ then $\psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi)) \in FL(\theta)$.
- If $\varphi \mathcal{R} \psi \in FL(\theta)$ then $\psi \wedge (\varphi \vee \bigcirc(\varphi \mathcal{R} \psi)) \in FL(\theta)$.
- If $\bigcirc\varphi \in FL(\theta)$ then $\varphi \in FL(\theta)$.

Example:

Let $\theta = p \,\mathcal{U} \neg p$. Then

$$FL(\theta) = \{p \,\mathcal{U} \neg p,\ \neg p \vee (p \wedge \bigcirc(p \,\mathcal{U} \neg p)),\ \neg p,\ p \wedge \bigcirc(p \,\mathcal{U} \neg p),\ p,\ \bigcirc(p \,\mathcal{U} \neg p)\}$$

- Definition of Fischer-Ladner closure purely syntactical
- Following definition yields subsets of it closed under "satisfaction of subformulas" ("what else has to hold")

$\leadsto$ If $\varphi \vee \psi \in M$ then $\varphi \in M$ or $\psi \in M$.

Single out those sets that do not contain contradictions p and $\neg p$.

<u>Definition (Hintikka set)</u>:

Let Θ an LTL formula in positive normal form.

A <u>Hintikka set</u> for Θ is a subset $M \subseteq FL(\Theta)$ so that the following closure properties hold:

- $\varphi \vee \psi \in M$ implies $\varphi \in M$ or $\psi \in M$
- $\varphi \wedge \psi \in M$ implies $\varphi \in M$ and $\psi \in M$
- $\varphi \mathcal{U} \psi \in M$ implies $\psi \in M$ or ($\varphi \in M$ and $\bigcirc(\varphi \mathcal{U} \psi) \in M$)
- $\varphi \mathcal{R} \psi \in M$ implies $\psi \in M$ and ($\varphi \in M$ or $\bigcirc(\varphi \mathcal{R} \psi) \in M$).

A Hintikka set $M \subseteq FL(\Theta)$ is called <u>consistent</u> if there is no $p \in \mathcal{P}$ with $\{p, \neg p\} \subseteq M$.

Denote by $\mathcal{H}(\Theta)$ the set of <u>all consistent Hintikka sets</u> for Θ.

Define

$\mathcal{P}^+(M) := M \cap \mathcal{P}$ // set of propositions that occur <u>positively</u> in M.

$\mathcal{P}^-(M) := \{p \in \mathcal{P} \mid \neg p \in M\}$ // set of propositions that occur <u>negatively</u> in M.

<u>Example (continued)</u>:

Let $\Theta = p \mathcal{U} \neg p$.

Then $\mathcal{H}(\Theta) = \{\emptyset, \{p\}, \{\neg p\}, \{p, \bigcirc(p \mathcal{U} \neg p)\}, \{p \mathcal{U} \neg p, p, \bigcirc(p \mathcal{U} \neg p)\}, \dots\}$