Advanced Automata Theory
Exercise Sheet 4

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Out: May 2 Due: **May 8, 12:00**

Exercise 1: Presburger formulas & Parikh images

a) Present a Presburger formula φ such that every bound variable occurs in **precisely** one atomic expression and such that

$$Sol(\varphi) = \left\{ \begin{pmatrix} 2n+1 \\ n+3 \end{pmatrix} \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ \begin{pmatrix} 3n+1 \\ 2n+2 \end{pmatrix} \;\middle|\; n \in \mathbb{N} \right\}.$$

b) The **Parikh image** $\Psi : \Sigma^* \to \mathbb{N}^\Sigma$ mapps each word w to the vector $\Psi(w)$, where the components store the number of occurrences of each letter in w. For a language $\mathcal{L} \subseteq \Sigma^*$, let $\Psi(\mathcal{L}) = \{\Psi(w) \mid w \in \mathcal{L}\}$. For example for $\Sigma = \{a, b, c\}$:

$$\Psi(ababcb) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \Psi((aa)^*(bbb)^*) = \left\{ \begin{pmatrix} 2n \\ 3m \\ 0 \end{pmatrix} \;\middle|\; n, m \in \mathbb{N} \right\}.$$

Give an NFA A so that $\Psi(\mathcal{L}(A)) = Sol(\varphi)$ for the Presburger formula φ from a).

Exercise 2: "Presburger $\Rightarrow$ NFA"-Algorithm

a) Prove the correctness of the construction given in class:
For every $q \in \mathbb{Z}$ and $w \in (\mathbb{B}^n)^*$, the automaton accepts w starting from q
iff w encodes $\vec{c}$ with $\vec{a}\,\vec{c} \leq q$.

b) Construct a finite automaton over $\mathbb{B}$ for the atomic Presburger formula $x - 3y \leq 1$.

Exercise 3: "Presburger $\Rightarrow$ NFA" for atomic formulas with equality

One can modify the algorithm for $\vec{a}\,\vec{x} \leq b$ to produce an NFA for $\vec{a}\,\vec{x} = b$ by making the state $0 \in \mathbb{Z}$ the only accepting state and by changing the transition relation so that a transition

$$q \xrightarrow{\vec{\beta}} \frac{1}{2}(q - \vec{a}\,\vec{\beta})$$

is only added if $q - \vec{a}\,\vec{\beta}$ is even.

a) Use the modified algorithm to construct a finite automaton for $x - 2y = 1$.

b) Verify your result in a) by checking that

$$\mathcal{L}(A_{x-2y=1}) = \mathcal{L}(A_{x-2y\leq 1}) \cap \mathcal{L}(A_{-x+2y\leq -1}) .$$