Prof Roland Meyer	**Advanced Automata Theory** **Exercise Sheet 5**	TU Braunschweig
Dr Prakash Saivasan		Summer term 2017

Out: May 10 Due: May 15, 12:00

Exercise 1: Quantifier Elimination

a) Eliminate the quantifiers of the following formula using the method described in class:

$$\neg\forall x\colon 3x < 2y \vee y < 2x\ .$$

b) An existential Presburger formula is a formula as described by the following EBNF:

$$\varphi ::= t_1 < t_2 \mid t_1 = t_2 \mid \exists x\colon \varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2\ .$$

In particular, it does not contain negations and it only contains the atomic predicates $<$ and $=$.

Prove that existential Presburger formulas are equally expressive as Presburger formulas.

Hint: Use Presburger's theorem.

Exercise 2: Parikh Images of Regular Languages

a) Prove that if $\mathcal{L} \in \mathrm{REG}_\Sigma$ is regular, $\Psi(\mathcal{L})$ is semilinear.

b) Prove that for each semilinear set $S \subseteq \mathbb{N}^d$, there is a regular language $\mathcal{L}$ over $\Sigma = \{a_1, ..., a_d\}$ with $S = \Psi(\mathcal{L})$.

c) Let $\mathit{lsbf}^{\ -1} : (\mathbb{B}^d)^* \to \mathbb{N}^d$ be the map that takes a word of binary vectors w and returns the vector of natural numbers $\vec{n}$ such that w is the "least significant bit first"-encoding of $\vec{n}$.

Present a regular language $\mathcal{L}$ over $\mathbb{B}^d$ such that

$$\left\{ \mathit{lsbf}^{\ -1}(w) \;\middle|\; w \in \mathcal{L} \right\} \subseteq \mathbb{N}^d$$

is not semilinear.

Exercise 3: Parikh Images of Context Free Languages

Use Parkikh's theorem to compute a representation for the semilinear set $\Psi(\mathcal{L}(G))$ for the grammar G which has the rules:

a) $S \to ab \mid XZ,\ \ Z \to SY,\ X \to a,\ \ Y \to b$

b) $S \to XY \mid \varepsilon,\ \ X \to aSb,\ \ Y \to bSc$

Exercise 4: Closure Properties of Semilinear Sets

a) Let $S = \bigcup_{i\in\{1,\ldots,k\}} \mathcal{L}(c_i, P_i) \subseteq \mathbb{N}^d$ be semilinear. Prove that semilinear sets are closed under Kleene iteration:

$$\{v_1 + \ldots + v_t \mid t \in \mathbb{N} \text{ and } v_1, \ldots, v_t \in S\} = \bigcup_{J\subseteq\{1,\ldots,k\}} \mathcal{L}\big(\sum_{i\in J} c_i, \bigcup_{i\in J} P_i \cup \{c_i\}\big) \ .$$

b) To prove that semilinear sets are closed under intersection, we showed that the intersection of two linear sets $\mathcal{L}(c, \{u_1, \ldots, u_m\})$ and $\mathcal{L}(d, \{v_1, ..., v_n\}) \subseteq \mathbb{N}^d$ is semilinear. We defined

$$B = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{N}^{m+n} \middle| \sum_{i=1}^{m} x_i u_i = \sum_{i=1}^{n} y_i v_i \right\} \ .$$

Let s_B be the set of minimal elements of $B \setminus \{0\}$ with respect to the product order $\leq^d$, i.e. $u \leq^d v$ if $u_c \leq v_c$ for all $c \in \{1, \ldots, d\}$.

Prove that $B = \mathcal{L}(0, s_B)$.

Hint: You may use that $\leq^d$ is well-founded.