Advanced Automata Theory
Exercise Sheet 8

Out: June 1 Due: June 12, 10:00

Exercise 1: NBA and S1S

MSO[$<$, succ] is the monadic second order logic interpreted on ω-words in the expected way. Its (clearly equiexpressive) fragment MSO[succ] is commonly known as S1S, the (monadic) second order logic of one successor.

a) Define a S1S formula Inf(X) so that $S_w, I \models \text{Inf(X)}$ iff $I(X)$ is an infinite set.

b) Büchi's theorem (I) can be adapted to show that every NBA-definable language is S1S-definable. Illustrate the main ingredients needed to adapt Büchi's proof.

c) Büchi's theorem (II) can be adapted to show that every S1S-definable language is NBA-definable. Illustrate the main ingredients needed to adapt Büchi's proof.

Exercise 2: LTL

a) Show that every LTL-definable language is FO[$<$]-definable.[1]

b) EF-games and the EF-theorem remain valid for ω-languages too.
Making use of this fact, show that $(a\{a,b\})^\omega$ is not LTL-definable.

c) Recall that the regular language $(aa)^*$ is **not** FO-definable. Why do we need at least two letters in the alphabet, to separate FO in the ω-languages case?

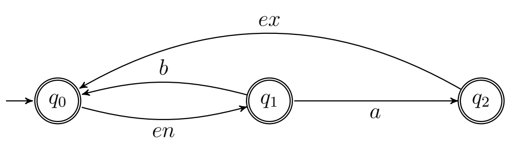
Exercise 3: Fairness

We define three notions of fairness (*en* and *ex* stand for "enabled" and "executed"):

Absolute fairness (impartiality):	$\Box\Diamond\, ex$	(**AF**)
Strong fairness (compassion):	$\Box\Diamond\, en \rightarrow \Box\Diamond\, ex$	(**SF**)
Weak fairness (justice):	$\Diamond\Box\, en \rightarrow \Box\Diamond\, ex$	(**WF**)

Which of the following statements hold for the NBA A depicted below?

$A \models \textbf{AF} \rightarrow \Box\Diamond a$ $\qquad A \models \textbf{SF} \rightarrow \Box\Diamond a$ $\qquad A \models \textbf{WF} \rightarrow \Box\Diamond a$



[1]here FO[$<$] is the first order fragment of MSO[$<$] over ω-words

Exercise 4: Unrollings

Prove the following equivalences:

$$\text{(a)}\quad \varphi\,\mathcal{U}\,\psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi\,\mathcal{U}\,\psi)) \qquad\qquad \text{(b)}\quad \varphi\,\mathcal{R}\,\psi \equiv \psi \wedge (\varphi \vee \bigcirc(\varphi\,\mathcal{R}\,\psi))$$

Exercise 5: LTL to GNBA

Consider the following specification of your life: $\varphi := \Box(study\,\mathcal{U}\,exam)$

a) Translate φ into an equivalent formula φ' that uses only the basic definition of LTL.

b) Translate φ' into an equivalent formula φ'' in PNF.

c) Compute the Fisher-Ladner closure $FL(\varphi'')$.

d) Using the method from class, compute a GNBA A_φ with $L(A_\varphi) = L(\varphi)$.

Exercise 6: Past Time LTL I

LTL as presented in the lecture reasons about the future, i.e. positions right of the current one. We extend LTL to also reason about the past, i.e. positions left of the current one. The full syntax of Past Time LTL is as follows:

$$\varphi \quad ::= \quad p \mid \varphi \vee \psi \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi\,\mathcal{U}\,\psi \mid \underbrace{\ominus\varphi}_{\text{"previous"}} \mid \underbrace{\varphi\,\mathcal{S}\,\psi}_{\text{"since"}}$$

The two new operators are defined as follows:

- $w, i \models \ominus\varphi$ iff $i > 0$ and $w, i-1 \models \varphi$ *(i.e. the previous position satisfies φ)*
- $w, i \models \varphi\,\mathcal{S}\,\psi$ iff there is $k \leq i$ so that for all $k \leq j < i$ we have $w, j \models \varphi$, and $w, k \models \psi$ *(i.e. there is a position where ψ holds and since then φ was satisfied until now)*

We define LTL[. . .] as the LTL syntax restricted to $p, \vee, \neg$ and the operators given in brackets. Note that we can always use $\wedge$ and $\rightarrow$ because they can be expressed using the included operators.

a) Find LTL[$\ominus, \mathcal{S}$] definitions for $\diamondminus$ and $\boxminus$ such that

1. $w, i \models \diamondminus\varphi$ iff there is $k \leq i$ with $w, k \models \varphi$
2. $w, i \models \boxminus\varphi$ iff $w, k \models \varphi$ holds for all $k \leq i$

b) The formula $\Box(ack \rightarrow \diamondminus req)$ specifies that every acknowledgment is preceded by an earlier request. Give an equivalent LTL[$\bigcirc, \mathcal{U}$] formula.

c) Kamp's theorem states the following:

> A language is FO[<] definable if and only if it is LTL[$\bigcirc, \mathcal{U}$] definable.

Using Kamp's theorem, show that a language is LTL[$\bigcirc, \mathcal{U}$] definable iff it is LTL[$\bigcirc, \mathcal{U}, \ominus, \mathcal{S}$] definable.

Exercise 7: Past Time LTL II

Let $\models_{\text{fin}}$ be the satisfaction relation between **finite** words and LTL[$\bigcirc, \mathcal{U}, \ominus, \mathcal{S}$] formulas, defined exactly as the $\models$ relation but only considering positions of the word at hand. We further define $\mathcal{L}_{\text{fin}}(\varphi) = \{u \in \Sigma^* \mid u, 0 \models_{\text{fin}} \varphi\}$.

We want to show that properties of the shape $\Diamond \varphi$ where φ only talks about the past can only speak about finite prefixes (i.e. they are safety properties). Consider $\varphi \in$ LTL[$\ominus, \mathcal{S}$]:

a) Prove that, for every $w \in \Sigma^\omega$, every position $i \in \mathbb{N}$ and every $j \geq i$,

$$w, i \models \varphi \qquad \text{iff} \qquad w_0 \dots w_j, i \models_{\text{fin}} \varphi$$

b) Prove $\mathcal{L}_{\text{fin}}(\Diamond \varphi)$ is a star-free language.

c) Show that $\mathcal{L}(\Diamond \varphi) = R \cdot \Sigma^\omega$ for some star-free language $R \subseteq \Sigma^*$.

[*Hint: try with* $R = \mathcal{L}_{\mathit{fin}}(\Diamond \varphi)$]