Proof ($\equiv_k$ is of finite index):

Consider finite set of variables V.

For every $k \in \mathbb{N}$ there are, up to logical equivalence, finitely many formulas of $qd = k$ (over V).

Proof:

By induction on qd.

Base case: Sufficient to consider formulas in DNF

$qd = 0$.

$$\bigvee_j \underbrace{(\bigwedge c_{je})}_{} \to \text{Disjunct.}$$

Each disjunct is a conjunction of

↳ atomic propositions $c = P_a(x)$ and $c = x<y$.

↳ and their negations.

Estimate their number:

↳ $P_a(x)$ with $x \in V$ $\rightsquigarrow$ $|\Sigma||V|$

↳ $x<y$ with $x, y \in V$ $\rightsquigarrow$ $|V|^2$

Together with negations $\rightsquigarrow$ $2(|\Sigma||V| + |V|^2)$.

Hence, number of disjuncts bounded by

$$2^{2(|\Sigma||V| + |V|^2)}$$

Hence, number of DNFs bounded by

$$2^{2^{2(|\Sigma||V| + |V|^2)}}$$

Induction step:

similar.

- Every equivalence class of $\equiv_k$ is determined by formulas φ that hold (with $qd(\varphi) \le k$ and n-free variables).
- There are, up to logical equivalence, $p_i \in \mathbb{N}$ formulas of $qd = i$.
- Hence, there are at most $2^{(p_0 + \dots + p_k)}$ classes. □