Positional Determinacy

Know: In every position, at most one player has a winning strategy:

$W_{A} \cap W_{P} = \emptyset$.

Now: Determinacy

In every position, either A or P has a winning strategy

↳ Follows from a more general result: Martin's theorem

↳ Give a direct proof (due to Zielonka, shows positional strategies)

↳ Result is not obvious:

why should one player have a winning strategy if the other does not

↳ Moreover, positional strategies are particular to parity games. Does not hold for Muller games.

Definition ($G|_{Pos'}$):

Let $G = (Pos_A, Pos_P, \rightarrow, \Omega)$ be a parity game and $Pos' \subseteq Pos$.

Then $G|_{Pos'} ::= (Pos_A \cap Pos', Pos_P \cap Pos', \rightarrow \cap Pos' \times Pos', \Omega|_{Pos'})$.

If $G|_{Pos'}$ is deadlock-free, call it a subgame of G.

Theorem (Determinacy):

Let $G = (Pos_A, Pos_P, \rightarrow, \Omega)$ be a parity game.

We have $W_A \cup W_P = Pos$.

Moreover, there are positional winning strategies for both, A and P.

Proof (Zielonka, LICS?, '98)

Induction on the highest priority in G.

IA ($n=0$): Any strategy for A is a winning strategy from all positions.

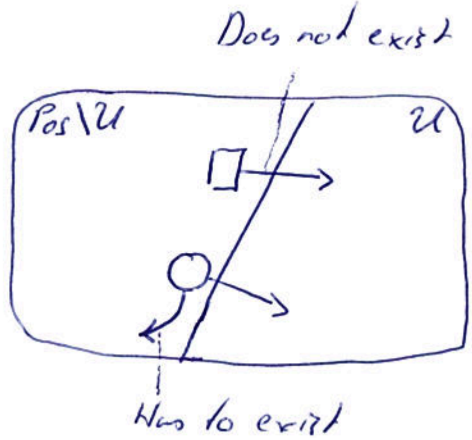
Moreover, each position has a successor so that such a strategy really exists.
Choose a positional such strategy.
Now

$$Pos = W_A = W_A \cup \emptyset = W_A \cup W_P.$$

<u>IS:</u> Consider the highest priority $n > 0$
Assume that n is even,
otherwise we change the roles of A and P in the argumentation.

↳ Let $\mathcal{U}$ be the set of positions from which P has a positional winning strategy.
↳ By the lemma from last lecture there is even a <u>single</u> strategy for player P that is winning from <u>all</u> positions in $\mathcal{U}$.

<u>We show:</u>
On all positions $Pos \setminus \mathcal{U}$, player A has a positional winning strategy.



<u>Claim: $G|_{Pos\setminus\mathcal{U}}$ is a game</u>
↳ • In G, there is no move for player P from $p \in Pos_P \setminus \mathcal{U}$ to $p' \in \mathcal{U}$.
• Otherwise p would be part of $\mathcal{U}$ as player P would have a positional winning strategy from there.
↳ • Moreover, player A is not deadlocked in $G|_{Pos\setminus\mathcal{U}}$.
• If this was the case, then in G there was $p \in Pos_A \setminus \mathcal{U}$ so that for all $p' \in Pos$ with $p \to p'$ we have $p' \in \mathcal{U}$
• But this would mean also $p \in \mathcal{U}$.

Case 1: The highest priority n does not occur in $G|_{Pos\setminus\mathcal{U}}$.

By the induction hypothesis,

$Pos\setminus\mathcal{U} = W_A \cup W_P$ so that

- A has a positional winning strategy from positions in W_A
- P has a positional winning strategy from positions in W_P.

Show that A not only wins $G|_{Pos\setminus\mathcal{U}}$ but also G:

↳ Clearly the case as P cannot leave $Pos\setminus\mathcal{U}$.

Show that $W_P = \emptyset$ and hence $W_A = Pos\setminus\mathcal{U}$:

↳ Assume there is $p \in W_P$, the set of winning positions for P in the game $G|_{Pos\setminus\mathcal{U}}$.

↳ Now from p, player P would win the overall game G, not only $G|_{Pos\setminus\mathcal{U}}$.

Case 1: A stays in $Pos\setminus\mathcal{U}$:

↳ Then P wins with positional winning strategy for $G|_{Pos\setminus\mathcal{U}}$.

Case 2: A moves to $\mathcal{U}$

↳ Then P wins with the positional winning strategy it has there.

↳ Hence, $p \in Pos\setminus\mathcal{U}$, but also $p \in \mathcal{U}$ because player P wins G from there. ↯ $W_P = \emptyset$.

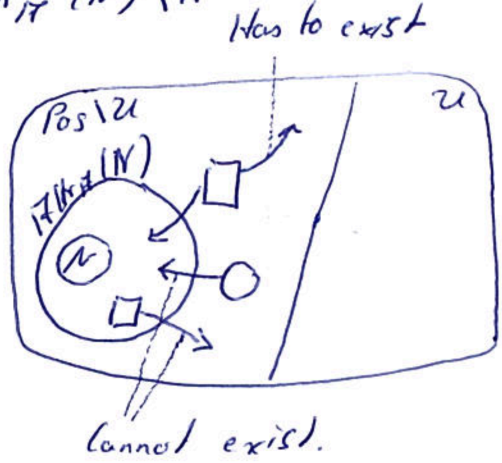
Case 2: The highest priority n does occur in $G|_{Pos\setminus\mathcal{U}}$:

↳ Define

$N := \{ p \in Pos\setminus\mathcal{U} \mid \Omega(p) = n \}$

↳ Consider $Attr_A(N)$, the set of positions from which A can force a visit of N.

Then
- there is no A move into $Attr_A(N)$
- there is no P move out of $Attr_A(N) \setminus N$
- $N \subseteq Attr_A(N)$



Consider $G|_Z$ with $Z = (Pos \setminus U) \setminus Attr_A(N)$

Show $G|_Z$ is a game, i.e., deadlock-free

If for $p \in Pos_P$ in Z
all moves lead to $Attr_A(N)$,
then p was in $Attr_A(N)$.

Apply induction hypothesis to $G|_Z$:

- Gives
$$Z = W_A \cup W_P$$
- Moreover, there are positional strategies for A and P.

Show that $W_P = \emptyset$ and hence $U_A = Z$:

- If $p \in W_P$, then the strategy not only wins $G|_Z$ but the full game G.
  - $\Rightarrow$ A cannot move to $Attr_A(N)$, there are no such moves.
  - $\Rightarrow$ If A moves to U, P plays the winning strategy there.
- Hence, $p \in U$ because p is winning for G
- Moreover, p is chosen from $(Pos \setminus U) \setminus Attr_A(N)$.

$\lightning$ Such a p cannot exist, $W_P = \emptyset$.

Show that A wins on W_A and on $Attr_A(N)$ the whole game G

Strategy:

On W_A: play winning strategy for $G|_Z$

On $Attr_A(N)$: force visit of N

Note that in both cases, we do not leave $Pos \setminus U$.

Why is the strategy winning?

<u>1. Case</u>: play leads to $Attr_{\forall}(N)$, and hence N, infinitely often

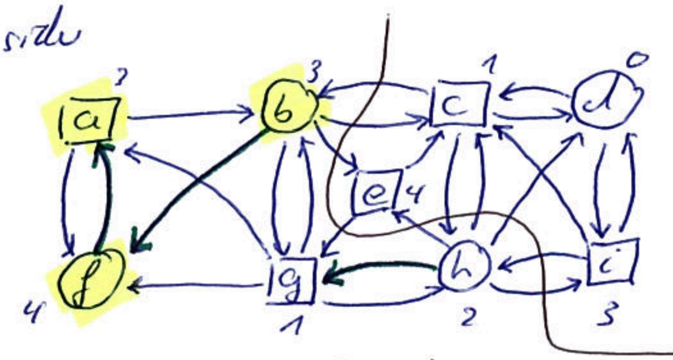
$\Rightarrow$ $\forall$ wins as n is maximal and even.

<u>2. Case</u>: play eventually stays outside $Attr_{\forall}(N)$

$\Rightarrow$ $\forall$ wins by induction hypothesis for $G|_Z$. $\square$

The induction step - on an example:

Consider



- P has a positional winning strategy on $U = \{c, d, i, e\}$ by assumption.
- Show that $\forall$ has a positional winning strategy on the remaining positions.
- Consider the positions with the highest priority. Here, $N = \{f\}$.
- Compute $Attr_{\forall}(N) = \{a, b, f\}$.
- The remaining game is played on $Z = \{g, h\}$.
- The induction hypothesis yields winning regions for $\forall$ and P.

  In the example, $W_{\forall} = \{g, h\}$, $W_P = \emptyset$.
- Compose strategy for $G|_Z$ with attractor strategy

  $\Rightarrow$ Yields positional winning strategy $\rightarrow$ for $Pos \setminus U$.