Computing the Deligne number of curve singularities and an algorithmic framework for differential algebras in SINGULAR

Master's Thesis

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Abstract

Let $S = \mathbb{C}\{x\}/I$ be a curve singularity, where I is defined by polynomials. The *Deligne number* of S is defined by $3\delta - m$, where δ is the *delta invariant* and m is the length of $\operatorname{Der}_{\mathbb{C}}(\overline{S})/\operatorname{Der}_{\mathbb{C}}(S)$. In this master's thesis we show that we can reduce the computation of this and other invariants of S to computations in $\mathbb{Q}[x]_{\langle x\rangle}$. To achieve this, we first show that integral closure, Kähler differentials and derivation modules behave well with respect to field extension and completion. Using that the length of a module only changes up to a factor under flat extensions, we are able to prove that the length does not change at all under the aforementioned operations. Finally, we will prove stability results, stating that the the *delta invariant*, the *multiplicity of the conductor* and the *Deligne number* do not change under field extension and completion. Using this, we will see that the mentioned invariants keep their values if we pass from $Q[x]_{\langle x\rangle}/I$ to S.
We will also state algorithms for the invariants. These can be applied over $\mathbb{Q}[x]_{\langle x\rangle}$ and taking the stability results into account, we can deduce that we are able to effectively compute the 3 invariants for curve singularities.

In addition, we will also offer a Singular implementation of differential algebras. We state results which allow us to represent these algebras and their elements and we provide a user-friendly and intuitive environment for computing with these objects.

Statutory declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources and that I have marked all material which has been quoted from the used sources.

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Introduction

In the study of curve singularities, there were several approaches to characterize quasi-homogeneity: In 1966, O. Zariski showed that quasi-homogeneous plane curves are those curves whose module of differentials have maximal torsion, see [Zar66]. Another differential approach is due to K. Saito in 1971, see [Sai71]. He proved that a hypersurface singularity $R = \mathbb{C}\{x\}/\langle f\rangle$ is quasi-homogeneous if and only if there is a derivation $\delta : R \to R$ that preserves f, which means: $\delta(f) = f$. He also proves the equivalence to a third fact: the exactness of the Poincaré complex. In 1977, G. Scheja and H. Wiebe showed that for a reduced complete intersection singularity R, the following equivalence holds: R is quasi-homogeneous if and only if there is an epimorphism from the module of differentials to the maximal ideal of R, see [SW77]. This was generalized by E. Kunz and W. Ruppert to arbitrary reduced curve singularities in the same year, see [KR77]. A numerical characterization was found by G.-M. Greuel in 1982, see [Gre82]. He stated that an irreducible Gorenstein singularity is quasi-homogeneous if and only if the *Deligne number* e and the *Milnor number* μ coincide.
The *Deligne number* of a curve singularity is named after P. Deligne. In his work [Mr0, Exposé X] from 1973, the number arose naturally as dimension of a smoothing component of the semiuniversal base: if R is a reduced smoothable curve singularity and E a smoothing component, then $\dim E = 3\delta - m$, where δ denotes the *delta invariant* and $m = \operatorname{length}_R(\operatorname{Der}_{\mathbb{C}}(\overline{R})/\operatorname{Der}_{\mathbb{C}}(R))$, see [Mr0, Theorem 2.27]. The embedding $\operatorname{Der}_{\mathbb{C}}(R) \hookrightarrow \operatorname{Der}_{\mathbb{C}}(\overline{R})$ goes back to a result of Seidenberg. In 1966, he has proven that there is an inclusion $\operatorname{Der}_{\mathbb{C}}(R) \hookrightarrow \operatorname{Der}_{\mathbb{C}}(\overline{R})$ in the case, where R is an integral domain, see [Sei66]. Since R is reduced, the normalization $\overline{R}$ is a product of normalizations of integral domains: $\overline{R} = \prod \overline{R/P}$, where P runs over all branches of R. One can show that the module $\operatorname{Der}_{\mathbb{C}}(R)$ embeds into $\prod \operatorname{Der}_{\mathbb{C}}(R/P)$ and with the result of Seidenberg, $\prod \operatorname{Der}_{\mathbb{C}}(R/P)$ embeds into $\prod \operatorname{Der}_{\mathbb{C}}(\overline{R/P}) = \operatorname{Der}_{\mathbb{C}}(\overline{R})$.
Although Deligne‘s formula is a local statement, he used global techniques to show it. The first local proof of the formula was found by G.-M. Greuel and E. Looijenga in 1984. They proved a conjecture of J. Wahl ([Wah81]) about the dimension of a smoothing component of a complex-analytic germ with isolated singularity, see [GL85]. In the article, they were also able to show that their result is a generalization of Deligne‘s formula and they deduced a local proof of it.
We will define the *Deligne number* to be $e = 3\delta - m$ like in [Gre82]. G.-M. Greuel did not only state the aforementioned numerical criterion for quasi-homogeneity in this paper, he also related the *Deligne number* to other invariants of R and gave many equivalent formulas for it. It was even possible to establish a link between the *Deligne number* and the torsion module of $\widetilde{\Omega}^1_{R/\mathbb{C}}$, the universally finite module of differentials of R. This idea was then used by by G.-M. Greuel, B. Martin and G. Pfister in 1985 to generalize the

numerical criterion to reduced Gorenstein singularities, see [GMP85]. They also proved a third equivalent description of quasi-homogeneity which is a generalization of the criterion given in [Zar66]: R is quasi-homogeneous if and only if the module of differentials of R has maximal torsion.
We can deduce that computing the *Deligne number* is of particular interest; we could decide if Gorenstein curve singularities are quasi-homogeneous and we could relate the number to other invariants, as in [Gre82]. But we need to state an algorithm to compute this number, and other invariants, and we need to argue why we can compute over the convergent power series ring $\mathbb{C}\{x\}$. In fact, we will state algorithms for the *delta invariant*, the *multiplicity of the conductor* and the *Deligne number* and we will reduce computations over $\mathbb{C}\{x\}$ to computations over $\mathbb{Q}[x]_{\langle x \rangle}$ using flatness properties. If $R = \mathbb{C}\{x\}/I$, where I is defined by polynomials, then we can perform the following flat extensions of $\mathbb{Q}[x]_{\langle x \rangle}/I$: first, we extend the ground field from $\mathbb{Q}$ to $\mathbb{C}$. After that, we take the $\langle x \rangle$-adic completion and altogether we get the flat extensions:

$$\mathbb{Q}[x]_{\langle x \rangle}/I \to \mathbb{C}[x]_{\langle x \rangle}/I\mathbb{C}[x]_{\langle x \rangle} \to \mathbb{C}[[x]]/I\mathbb{C}[[x]]$$

On the other hand, we can also take the $\langle x \rangle$-adic completion of R and obtain the flat extension:

$$\mathbb{C}\{x\}/I \to \mathbb{C}[[x]]/I\mathbb{C}[[x]]$$

During the thesis we will prove stability results, stating that the mentioned invariants do not change under field extension and completion. Hence, we can compute the invariants over the ring $\mathbb{Q}[x]_{\langle x \rangle}/I$ and they keep their value when passing to R.
The basis of this consideration is established in the first chapters of this thesis. In Chapter 1, we give a short introduction to associated primes, reduced rings, separable field extensions and completion. It can be seen as a collection of properties which we will use over and over again. Chapter 2 treats the normalization of reduced rings. We will give the basic definitions and properties, argue when the normalization is finite as a module and reason about normal rings. In addition, we give an introduction to excellent rings since their normalization and completion commute, which is useful when we prove stability results concerning completion. We will also prove a theorem which allows us to commute field extension and integral closure. This will be helpful in the proofs of stability results concerning field extensions. Chapter 3 considers Kähler differentials, differential algebras, derivations and universally finite differentials. We will show that these notions partly commute with field extension and completion. In Chapter 4, we first prove results, stating that the length of modules is preserved under field extension and completion. Then we define the aforementioned invariants and prove the stability results. We also establish some algorithmic ideas to compute the invariants.

Algorithms are considered in Chapter 5. First, we state some basic algorithms of rings and modules that we will need, then we give a short introduction to normalization algorithms of Grauert-Remmert type and finally, we state the algorithms for computing the invariants. We will also consider some examples of quasi-homogeneous and non-quasi-homogeneous curve singularities and reason about their invariants.

The appendix of this thesis consists of three chapters. In A, we collect some basic results from several topics of commutative algebra. Chapter B gives insight into a SINGULAR implementation of the algorithms for computing the invariants. And in Chapter C, we give an introduction to an implementation of differential algebras using SINGULAR: the library `difform.lib`.

Table of Contents

1. Associated primes, reduced rings and completion

The first chapter of this thesis contains some basic notions from commutative algebra, that are needed in later chapters. In 1.1, we give a recall of what associated and minimal primes are. Section 1.2 gives the definition of reduced rings, states some basic properties and shows how finitely many minimal prime ideals influence the definition. A short excursion to separable field extensions and geometrically reduced algebras is given in 1.3: We state some properties and introduce the important notion of perfect fields. The last section, 1.4, is a recall of the completion of a module.

Motivated by geometry, the main theorems of this introduction are valid for reduced Noetherian algebras over fields of characteristic 0 (or perfect fields) - we will later focus on rings of this type.

1.1 Associated primes

Associated primes naturally arose as a generalization of the fundamental theorem of arithmetic. In more general rings than $\mathbb{Z}$, the UFD-property may fail and we are not longer able to decompose elements into products of primes. But if the considered ring is at least Noetherian, we still can represent an ideal I as intersections of *primary ideals*. This is called *primary decomposition*. The *associated primes* of I are then the radicals of the *primary ideals* appearing in the decomposition. We refer to [AM69, Chapter 4] for the theory of primary decomposition.
Of particular importance are *minimal associated primes*: these are *associated primes* that satisfy a certain minimality property over I. Such ideals will be helpful when dealing with *reduced rings* in Section 1.2 and we will use them to lift *derivations* to the *normalization* of a ring in Section 4.4. *Minimal associated primes* even have a geometric interpretation: they correspond to the branches of a variety.
In order to circumvent a restriction to ideals, we introduce *associated primes* for modules:

Definition 1.1.1. Let R be a ring and M an R-module. A prime ideal P of R is called **associated prime** of M if $P = \operatorname{Ann}_R(m)$ for an element $m \in M$. The set of all associated primes is denoted by $\operatorname{Ass}_R(M)$ or just $\operatorname{Ass}(M)$ if it is clear which ring is used.

Since the annihilator of an element $m \in M$ is the kernel of the multiplication map $R \to M$, $r \mapsto r \cdot m$, we get an alternative condition for a prime ideal to be associated to a module and we can collect some useful facts:

Remark 1.1.2. Let R be a ring and M an R-module.

a) A prime ideal P is an associated prime of M if and only if there is an R-linear injection $R/P \hookrightarrow M$.

b) If N is a submodule of M, then we have: $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$.

c) If P is a prime ideal of R, then $\operatorname{Ass}_R(R/P) = \{P\}$.

Proof. One direction in the proof of a) is clear by the above consideration. For the other direction, let $\varphi : R/P \hookrightarrow M$ be an injection. Denote by $m \in M$ the image of $\overline{1} \in R/P$, then we get for an element $x \in P$:

$$0 = \varphi(\overline{0}) = \varphi(\overline{x}) = x \cdot \varphi(\overline{1}) = x \cdot m$$

So x is in the annihilator of m. If we start with an element y from $\operatorname{Ann}_R(m)$, we can again use the R-linearity of φ:

$$0 = y \cdot m = y \cdot \varphi(\overline{1}) = \varphi(\overline{y})$$

The map is injective, hence $y \in P$. Altogether, $P = \operatorname{Ann}_R(m)$ and P is an associated prime of M.

The proof of b) is clear by the characterization given in a): any injective map $R/P \hookrightarrow N$ can be composed with $N \hookrightarrow M$ to an injection $R/P \hookrightarrow M$.

In order to prove part c), let P be a prime ideal of R. It is clear that P is an associated prime of R/P since we have the injective identity map from R/P to itself. Now let Q be an arbitrary associated prime ideal of R/P. Then we get an injection $\varphi : R/Q \hookrightarrow R/P$. We have to show the equality $P = Q$. Let $x \in P$, then we have:

$$\overline{0} = \overline{x} \cdot \varphi(\overline{1}) = \varphi(\overline{x})$$

Since φ is injective, x has to lie in Q. For the converse inclusion, let $x \in Q$. Then the equation

$$\overline{0} = \varphi(\overline{x}) = \overline{x} \cdot \varphi(\overline{1})$$

and the fact that R/P is an integral domain imply that either $\varphi(\overline{1})$ or $\overline{x}$ is $\overline{0}$. If the image of $\overline{1}$ under φ would be zero, then φ would be the zero map which is not injective. Hence, $\overline{x} = \overline{0}$ and $x \in P$. ■

Example 1.1.3.

a) Let p be a prime number in $\mathbb{Z}$. The associated prime ideals of $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-module can be easily determined using remark 1.1.2: $\mathrm{Ass}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) = \{p\mathbb{Z}\}$.

b) Let k be a field, $R = k[x, y]$ and $M = k[x, y]/\langle x^2, xy\rangle$. Then we show that $\mathrm{Ass}_R(M)$ consists of the prime ideals $\langle x\rangle$ and $\langle x, y\rangle$:

Proof. First, we prove that we can represent the two prime ideals as annihilator, hence they are associated to M.

The ideal $\langle x\rangle$ is the annihilator $\mathrm{Ann}_R(\overline{y})$:
The monomial x clearly annihilates $\overline{y}$, so the inclusion " $\subseteq$ " is clear. Let f be from $\mathrm{Ann}_R(\overline{y})$, then we have that $\overline{0} = \overline{fy}$ and thus: $fy \in \langle x^2, xy\rangle$. So we get a representation

$$\begin{aligned} fy &= gx^2 + hxy \\ \Leftrightarrow \underbrace{fy - hxy}_{\in\langle y\rangle} &= gx^2 \end{aligned}$$

and since $\langle y\rangle$ is a prime ideal in R and $x^2 \notin \langle y\rangle$, the polynomial g has to be in $\langle y\rangle$. But then we can rule out y on both sides in

$$fy = gx^2 + hxy$$

since R is an integral domain and obtain that $f \in \langle x\rangle$.
With the same kind of arguments, one can show that $\langle x, y\rangle = \mathrm{Ann}_R(\overline{x})$.

To complete the proof, we need to show that no other prime ideals are annihilators of elements of M. Assume there is a prime ideal P of R which is associated to M. Then $P = \mathrm{Ann}_R(\overline{f})$, where $\overline{f}$ is from M. We immediately get that $x^2 \in P$ since this annihilates any element in M. This implies $x \in P$ because P is prime. The ideal P annihilates $\overline{f}$, so $fP \subseteq \langle x^2, xy\rangle \subseteq \langle x\rangle$ and thus $f \in \langle x\rangle$ or $P \subseteq \langle x\rangle$. If $P \subseteq \langle x\rangle$, we would get that $P = \langle x\rangle$ and if $f \in \langle x\rangle$, then $y \in P$ and therefore $P = \langle x, y\rangle$. ∎

c) If R is an integral domain, then the only associated prime of R is $\langle 0\rangle$: Of course, $\langle 0\rangle = \mathrm{Ann}_R(1)$ and if P is a prime ideal that is the annihilator of an element $x \in R$, then $Px = 0$. So, either $x = 0$ or $P = 0$. But x cannot be 0 since then, $P = R$, which is a contradiction to P being a prime ideal.

By the Noetherian property of rings, we get a maximal element of any non-empty set of ideals. The following proposition is proven via this fact and it ensures a desirable behaviour of the associated primes of a module: they vanish if and only if the module itself vanishes. Also the finiteness, which is encoded in the Noetherian property, influences the associated primes. This can be seen in Lemma 1.1.5.

Proposition 1.1.4. *Let R be a Noetherian ring and M an R-module. Then $M = 0$ if and only if $\operatorname{Ass}(M) = \emptyset$.*

Proof. The proof can be found in [Mat80, Proposition 7.B, Corollary 1]. ■

Lemma 1.1.5. *If R is a Noetherian ring and M a finitely generated R-module, then $\operatorname{Ass}(M)$ is a finite set.*

Proof. This is [Stacks, Tag 00LC] ■

We know that under localization of a ring R at a multiplicatively closed set $S \subseteq R$, we get a one-to-one correspondence for prime ideals:

$$\begin{aligned} \{P \in \operatorname{Spec}(R) \,|\, P \cap S = \emptyset\} &\longrightarrow \operatorname{Spec}(S^{-1}R) \\ P &\longmapsto S^{-1}P \end{aligned}$$

It turns out, that a restriction of this map is a one-to-one correspondence on the associated primes:

Proposition 1.1.6. *Let R be a Noetherian ring, M an R-module and S a multiplicatively closed subset of R. Then there is a one-to-one correspondence:*

$$\begin{aligned} \{P \in \operatorname{Ass}_R(M) \,|\, P \cap S = \emptyset\} &\longrightarrow \operatorname{Ass}_{S^{-1}R}(S^{-1}M) \\ P &\longmapsto S^{-1}P \end{aligned}$$

Proof. A detailed proof is given in [AK70, Proposition 3.9]. ■

We want to relate the support of an R-module M and the associated primes of M to give a characterization of the minimal elements in $\operatorname{Ass}_R(M)$. So, focusing on a single prime ideal Q, we obtain by the above proposition:

$$\operatorname{Ass}_{R_Q}(M_Q) = \{P \in \operatorname{Ass}_R(M) | P \subseteq Q\}$$

Hence, using Proposition 1.1.4, we can precisely say, when the localization M_Q does not vanish: This happens if and only if the associated primes of M_Q are not empty and by the above identification, if and only if there is an associated prime P of M, lying in Q. We have therefore proven the following statement:

Corollary 1.1.7. *For a Noetherian ring R and an R-module M, we get an alternative description of the support of M:*

$$\operatorname{Supp}(M) = \bigcup_{P \in \operatorname{Ass}(M)} V(P)$$

Now we look at the minimal elements of the support of a module. With the above corollary, it turns out that these elements are not only associated primes but also *minimal associated primes*. But first, we have to introduce this notion:

Definition 1.1.8. Let R be a ring, $I \trianglelefteq R$ an ideal and M an R-module.

a) If R is Noetherian, then the minimal elements of $\operatorname{Ass}(M)$ are called **minimal (associated) primes of** M. The set of all minimal primes of M is denoted by $\operatorname{Min}(M)$.
Associated primes which are not minimal, are called **embedded primes**.

b) A prime ideal P of R is called **minimal over** I if there does not exist another prime ideal Q so that $I \subseteq Q \subsetneq P$.

Lemma 1.1.9. *Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal and M an R-module, then:*

a) The minimal elements of $\operatorname{Supp}(M)$ *are the minimal primes of M.*

b) If M is finitely generated, then the minimal primes of M are the prime ideals of R which are minimal over $\operatorname{Ann}_R(M)$.

c) The set $\operatorname{Min}_R(R/I)$ *consists of all prime ideals which are minimal over I. In particular:* $\operatorname{Min}(R) = \{P \in \operatorname{Spec}(R) \mid \nexists Q \in \operatorname{Spec}(R) : Q \subsetneq P\}$, *so the minimal primes of R are intuitively the smallest prime ideals.*

d) The map

$$\begin{aligned} \operatorname{Min}_R(R/I) &\longrightarrow \operatorname{Min}_{R/I}(R/I) \\ P &\longrightarrow \overline{P} \end{aligned}$$

is a bijection.

Proof. First, we use Corollary 1.1.7 to show that we only need the minimal primes to describe the support:

$$\operatorname{Supp}(M) = \bigcup_{P \in \operatorname{Min}(M)} V(P)$$

The inclusion "$\supseteq$" is clear, since $\operatorname{Min}(M) \subseteq \operatorname{Ass}(M)$. To show the other inclusion, start with any associated prime ideal Q. We find a minimal prime

P that is contained in Q. Hence, $V(Q) \subseteq V(P)$ and thus $\bigcup_{Q \in \mathrm{Ass}(M)} V(Q) \subseteq \bigcup_{P \in \mathrm{Min}(M)} V(P)$.
Using this identification of $\mathrm{Supp}(M)$, the claim of part a) is an easy consequence.
To show b), we use that $\mathrm{Supp}(M) = V(\mathrm{Ann}_R(M))$ by Lemma A.1.3. Then a prime P is a minimal prime of M if and only if it is minimal in $\mathrm{Supp}(M)$, by a) and this is the case if and only if P is minimal over the annihilator.
Part c) is a special case of b): The annihilator of R/I is I.
The map in d) comes from the bijection $\{P \in \mathrm{Spec}(R) \mid P \supseteq I\} \to \mathrm{Spec}(R/I)$. We combine this with the equivalence: A prime P is in $\mathrm{Min}_R(R/I)$ if and only if P is minimal over I by c). This is true if and only if $\overline{P}$ is minimal over $\overline{0}$ in R/I. Hence, again using c), we conclude that this happens if and only if $\overline{P} \in \mathrm{Min}_{R/I}(R/I)$. This gives the desired result. ■

The above lemma can be used to explicitly find the minimal primes of a given module or ring:

Example 1.1.10. Let k be a field and $R = k[x,y]/\langle xy \rangle$. Then $\mathrm{Min}_{k[x,y]}(R) = \{\langle x \rangle, \langle y \rangle\}$. This can be easily seen using the lemma: the ideals $\langle x \rangle$ and $\langle y \rangle$ are the only primes in $k[x,y]$ which are minimal over $\langle xy \rangle$. Hence, by 1.1.9 c), we get the equality.
If we also apply part d) of the lemma, we obtain: $\mathrm{Min}_R(R) = \left\{\overline{\langle x \rangle}, \overline{\langle y \rangle}\right\}$.

1.2 Reduced rings

Integral domains are rings without zero divisors and we know from commutative algebra that such rings have nice properties: their total ring of fractions is a field, they have exactly one associated prime, namely 0, and geometrically they correspond, as coordinate rings, to irreducible varieties. In order to describe varieties with more than one branch in geometry, we have to generalize rings on the algebraic side. To obtain a more general class of rings than that of integral domains, we do not forbid all zero divisors but only the *nilpotent* elements. This will form the class of *reduced* rings which is rather common but still has very useful properties: If R is a *reduced* ring, the total ring of fractions $Q(R)$ is no longer a field but a product of fields. *Reduced* rings have no embedded components, so $\mathrm{Ass}(R) = \mathrm{Min}(R)$, the set of zero divisors of R is the union of all minimal primes and the *normalization* of R is a product of *normalizations* of integral domains. We shall start with the definition and facts about *nilpotent* elements.

Definition 1.2.1. Let R be a ring and $r \in R$. If there is a natural number $n \geq 1$ so that $r^n = 0$, then r is called **nilpotent**. The set of all nilpotent elements is called **nilradical** of R and we denote it by $\mathfrak{N}(R)$.

It is clear that any nilpotent element is a zero divisor and that $\sqrt{0} = \mathfrak{N}(R)$. So the nilradical is a radical ideal in R and we can give an alternative representation of it:

Proposition 1.2.2. *For a ring R, the nilradical $\mathfrak{N}(R)$ is the intersection of all prime ideals.*

Proof. See [AM69, Proposition 1.8] for a proof. ∎

Because of this representation, the nilradical commutes with localization. This will be the reason why *being reduced* is a local property.

Lemma 1.2.3. *Let R be a ring, $\mathfrak{N}(R)$ its nilradical and $S \subseteq R$ be multiplicatively closed, then: $S^{-1}\mathfrak{N}(R) = \mathfrak{N}(S^{-1}R)$.*

Proof. The nilradical is the intersection of all prime ideals by Proposition 1.2.2. Using that localization commutes with finite intersections, we can deduce:

$$S^{-1}\mathfrak{N}(R) = S^{-1} \bigcap_{P \in \operatorname{Spec}(R)} P = \bigcap_{P \in \operatorname{Spec}(R)} S^{-1}P$$

If $P \cap S \neq \emptyset$, then $S^{-1}P = S^{-1}R$, so by applying Proposition 1.2.2 again, the intersection reduces to:

$$\bigcap_{P \in \operatorname{Spec}(R), P \cap S = \emptyset} S^{-1}P = \bigcap_{Q \in \operatorname{Spec}(S^{-1}R)} Q = \mathfrak{N}(S^{-1}R),$$

since $\operatorname{Spec}(S^{-1}R) = \left\{ S^{-1}P \middle| P \in \operatorname{Spec}(R),\ P \cap S = \emptyset \right\}$. ∎

Remark 1.2.4. Let R be a ring and $\mathfrak{N}(R)$ its nilradical. Suppose, $\mathfrak{N}(R)$ is finitely generated, then we can write $\mathfrak{N}(R) = \langle s_1, \ldots, s_m \rangle$. Any of the generators is nilpotent. Hence, there exist n_i so that $s_i^{n_i} = 0$ for any $i = 1, \ldots, m$. Set $n = m \cdot \max_{i=1,\ldots,m} n_i$. Then any element $x \in \mathfrak{N}(R)$ satisfies: $x^n = 0$. Thus, the nilradical itself is nilpotent:

$$\mathfrak{N}(R)^n = 0$$

Definition 1.2.5. A ring R is called **reduced** if it has no nilpotent elements different from 0.

One could also ask whether the nilradical $\mathfrak{N}(R)$ of a ring R is 0. This is clearly equivalent to R being reduced, since the nilradical is the set of all nilpotent elements.
The reduced ring R may have zero divisors since not all of them are nilpotent. Therefore, note that the total ring of fractions is in general only a ring and not a field: zero divisors of R are non-units in $Q(R)$. But at least we cannot have nilpotent elements in the total ring of fractions:

Lemma 1.2.6. *Let R be reduced and $R \subseteq S$ a ring extension so that $S \subseteq Q(R)$. Then S is reduced.*
In particular: $Q(R)$ is reduced.

Proof. We show that $Q(R)$ is reduced. Then we just use the fact that subrings of reduced rings are reduced and immediately get that S is reduced. Assume, we have an element $\frac{a}{b}$ in $Q(R)$ that is nilpotent. Then there exists an $n \geq 1$ so that: $0 = \frac{a^n}{b^n}$. Hence, there is a non-zero divisor $u \in R$ with $0 = ua^n$ which directly implies $a^n = 0$. The ring R is reduced, hence $a = 0$ and $\frac{a}{b} = 0$. ■

A reduced ring can always be obtained from an arbitrary ring R by constructing the quotient: $R/\mathfrak{N}(R)$. If $\overline{r}$ was nilpotent in $R/\mathfrak{N}(R)$, then there would be an $n \geq 1$ so that $r^n \in \mathfrak{N}(R)$. But then also $r \in \mathfrak{N}(R)$ since the nilradical is a radical ideal and finally $\overline{r} = \overline{0}$. In fact, we only used here that $\mathfrak{N}(R)$ is a radical ideal, so this proof works for any ideal I of R with $\sqrt{I} = I$ and with the same argument, the converse is also true:

Remark 1.2.7. If R is a ring and I an ideal in R, then R/I is reduced if and only if I is a radical ideal.

Before we continue listing more properties of reduced rings, we should throw a glance at some examples.

Example 1.2.8. Through the following examples, let R be an arbitrary ring and k be a field.

a) The ring $k[x, y]/\langle xy \rangle$ is not an integral domain since $\langle xy \rangle$ is not a prime ideal in $k[x, y]$. But it is a radical ideal:

$$\sqrt{\langle xy \rangle} = \sqrt{\langle x \rangle \cap \langle y \rangle} = \sqrt{\langle x \rangle} \cap \sqrt{\langle y \rangle} = \langle x \rangle \cap \langle y \rangle = \langle xy \rangle$$

We used here that taking the radical ideal commutes with finite intersections. By Remark 1.2.7, the ring $k[x, y]/\langle xy \rangle$ is reduced. Geometrically, it is the coordinate ring of a variety with two branches: the crossing of the coordinate axis in $\mathbb{A}^2_k$.

b) The quotient ring $R[x]/\langle x^2 \rangle$ is clearly not reduced.

c) For an indeterminate x over R, we have that R is reduced if and only if $R[x]$ is reduced.

Proof. If $R[x]$ is reduced, then we have immediately that R is reduced, since R is a subring of $R[x]$. For the converse direction let $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ be nilpotent. Then there exists a $k \geq 1$ so that $f^k = 0$. We show by induction that this already implies $a_l = 0$ for

$l = 0, \dots, n$ and hence, $f = 0$.
For the base case define $g = \sum_{i=1}^{n} a_i x^i$, then we can write:

$$0 = f^k = (a_0 + g)^k = \sum_{j=0}^{k} \binom{k}{j} a_0^j g^{k-j} = a_0^k + \sum_{j=0}^{k-1} \binom{k}{j} a_0^j g^{k-j}$$

The minimal degree of a monomial of $\sum_{j=0}^{k-1} \binom{k}{j} a_0^j g^{k-j}$ is 1, so we must have: $a_0^k = 0$ and since R is reduced, it follows that $a_0 = 0$.
For the case $l > 0$ we can assume $a_i = 0$ for $i = 1, \dots, l-1$. Thus, f has the form: $f = \sum_{i=l}^{n} a_i x^i$. Now set $f' = \sum_{i=0}^{n-l} a_{i+l} x^i = \sum_{i=l}^{n} a_i x^{i-l}$, then we obtain:

$$0 = f^k = (x^l \cdot f')^k = x^{lk} \cdot (f')^k$$

This product is 0 if and only if $(f')^k = 0$ since x is not a zero divisor in the polynomial ring. So f' is nilpotent and by applying the base case, we can deduce that $a_l = 0$. Altogether: $R[x]$ is reduced. ■

d) By inductively applying part c), we can even say that a ring R is reduced if and only if $R[x_1, \dots, x_n]$ is reduced.

For an integral domain R, we have seen in Example 1.1.3 that the only associated prime is the zero ideal. For a reduced ring, $\mathrm{Ass}(R)$ usually consists of more than one element. But we can state, that there are no embedded components:

Lemma 1.2.9. *Let R be a reduced ring. Then:* $\mathrm{Min}(R) = \mathrm{Ass}(R)$.

Proof. The inclusion $\mathrm{Ass}(R) \supseteq \mathrm{Min}(R)$ holds by definition of the minimal associated primes. For the other inclusion we may assume that there exists an associated prime Q which is not minimal. However, Q contains a minimal prime P. Since $Q = \mathrm{Ann}(x)$ for some $x \in R$, we have:

$$xQ = 0 \subseteq P$$

By $Q \nsubseteq P$, and because P is prime, we get: $x \in P \subseteq Q$. But then x is in its own annihilator and therefore $x^2 = 0$. Since R is reduced, x is finally 0 and $Q = R$, which is a contradiction to Q being a prime ideal. ■

It can always be tested locally if a given ring is reduced:

Lemma 1.2.10. *For a ring R, the following are equivalent:*

a) R is reduced

b) R_P is reduced for all $P \in \mathrm{Spec}(R)$

c) R_m is reduced for all maximal ideals m of R.

Proof. We go straight ahead and prove a) implies b). Since R is reduced, the nilradical of R is 0. Now localize at a prime ideal and use Lemma 1.2.3:

$$0 = \mathfrak{N}(R)_P = \mathfrak{N}(R_P)$$

Therefore, R_P is reduced.
The implication b) to c) is clear, so we only have to show c) to a): For any maximal ideal m of R, we have that $0 = \mathfrak{N}(R_m) = \mathfrak{N}(R)_m$, again by Lemma 1.2.3. But then $\mathfrak{N}(R)$ is also 0. Hence, R is reduced. ■

To prove some more of the promised properties from the introduction of this section, we may assume that we have rings with finitely many minimal primes. This is always fulfilled if the rings we consider are Noetherian, see Lemma 1.1.5.

Lemma 1.2.11. *For a reduced ring R with finitely many minimal primes $P_1, \ldots, P_r$ we have that:*

a) R_{P_i} is a field.

b) $Q(R/P_i) = R_{P_i}$

c) The union $\bigcup_{i=1}^r P_i$ is the set of zero divisors of R.

d) $Q(R) = \prod_{i=1}^r Q(R/P_i)$

Proof. We start with a): We know that $\mathrm{Spec}(R_{P_i})$ is in one-to-one correspondence to the prime ideals of R that are contained in P_i. Since this is a minimal prime ideal, R_{P_i} has only one prime ideal: $(P_i)_{P_i}$. By Lemma 1.2.10, R_{P_i} is reduced. Thus, $0 = \mathfrak{N}(R_{P_i}) = (P_i)_{P_i}$, as the nilradical is the intersection of all prime ideals via Proposition 1.2.2. Finally, 0 being a maximal ideal implies that R_{P_i} is a field.
Applying laws of localization, one can show b):

$$R_{P_i} = R_{P_i}/(P_i)_{P_i} = (R/P_i)_{P_i/P_i} = (R/P_i)_{\langle 0 \rangle} = Q(R/P_i)$$

Where the first equality comes from the fact, that R_{P_i} is a field by a).
For the proof of c), we refer to [AM69, Proposition 4.7].
If we use part c), we get by [Stacks, Tag 02LX] that $Q(R) = \prod_{i=1}^r R_{P_i}$ and by b) we obtain the desired equality. Hence, we have proven d). ■

Before we move on with another theoretical property on reduced rings, let us consider an example to illustrate the lemma:

Example 1.2.12. Let k be a field and $R = k[x,y]/\langle xy \rangle$. Then we know by Example 1.2.8 and 1.1.10 that R is reduced and that the minimal primes are $\overline{\langle x \rangle}$ and $\overline{\langle y \rangle}$. If we factor out these primes, we obtain: $R/\overline{\langle x \rangle} = k[y]$ and $R/\overline{\langle y \rangle} = k[x]$. Hence, taking Lemma 1.2.11 into account, we can conclude:

$$Q(R) = Q(k[x]) \times Q(k[y]) = k(x) \times k(y)$$

Now let us take a look at a localization of the total ring of fractions of a reduced ring R at a multiplicatively closed subset W of R. This will behave well - it *commutes* with taking fractions. In the case that R is an integral domain and $Q(R)$ is a field, this result is immediately clear: $Q(W^{-1}R) = Q(R) = W^{-1}Q(R)$.

Lemma 1.2.13. *Let R be a reduced ring with finitely many minimal primes. For a multiplicatively closed set W in R, we have:*

$$Q(W^{-1}R) = W^{-1}Q(R)$$

Proof. By Lemma 1.2.11 and the fact that localization commutes with direct sums/products, we obtain the equality:

$$W^{-1}Q(R) = \prod_{P \in \mathrm{Min}(R)} W^{-1}(R_P)$$

Now we treat two cases: If $P \cap W = \emptyset$, then $W \subseteq R \setminus P$ and localizing R_P at W does nothing: $W^{-1}(R_P) = R_P$. If $P \cap W \neq \emptyset$, then we get by Lemma 1.2.11:

$$\emptyset \neq P \cap W \subseteq \mathrm{Ann}_R(Q(R/P)) \cap W = \mathrm{Ann}_R(R_P) \cap W$$

Hence, localizing R_P at W is 0 by Lemma A.1.4. Altogether, we obtain:

$$W^{-1}Q(R) = \prod_{\substack{P \in \mathrm{Min}(R) \\ P \cap W = \emptyset}} R_P$$

On the other hand, the ring $W^{-1}R$ is reduced as a localization of a reduced ring, so there are no embedded components by Lemma 1.2.9. Using Proposition 1.1.6, we know that the map, sending P to $W^{-1}P$ is a one-to-one correspondence between:

$$\{P \mid P \in \mathrm{Min}(R),\, P \cap W = \emptyset\} \to \mathrm{Min}(W^{-1}R)$$

Applying again Lemma 1.2.11, we get:

$$Q(W^{-1}R) = \prod_{\substack{P \in \mathrm{Min}(R) \\ P \cap W = \emptyset}} (W^{-1}R)_{W^{-1}P} = \prod_{\substack{P \in \mathrm{Min}(R) \\ P \cap W = \emptyset}} R_P = W^{-1}Q(R)$$

■

1.3 Separable field extensions and geometrically reduced algebras

This section treats a generalization of separability for algebraic field extensions to arbitrary field extensions. This more common notion is needed in 2.5, when we lift the normalization of a k-algebra S to the L-algebra $S \otimes_k L$, where L is an extension field of k. The success of this will heavily depend on the ground field k - we need it to be *perfect*: any field extension of k is *separable*.
The second notion introduced here, is that of *geometrically reduced* algebras: reduced algebras which stay reduced when we extend the ground field. We will see that the concept of *separability* and *geometrically reduced* algebras coincide in the field case.

Definition 1.3.1. Let k be a field and L/k an arbitrary field extension.

a) We say L **is separably generated over** k if there exists a transcendence basis $\{x_i \mid i \in I\}$ of L/k so that the extension $k(x_i \mid i \in I) \subseteq L$ is separable algebraic.

b) The field L is called **separable over** k if for any intermediary field $k \subseteq K \subseteq L$ that is a finitely generated field extension of k, we have that K/k is separably generated.

c) k is called **perfect** if any field extension of k is separable.

Example 1.3.2. Let k denote a field.

a) If $\mathrm{char}(k) = 0$, then any extension is separable: Let L/k be an arbitrary field extension and $k \subseteq K \subseteq L$ a finitely generated intermediary extension of k. Then for any transcendence basis $\{x_i \mid i \in I\}$ of K/k, we obtain that $k(x_i \mid i \in I) \subseteq K$ is algebraic and since we act in characteristic 0, this is even separable algebraic.

b) Let now $\mathrm{char}(k) = p \neq 0$ and assume, there is an element a of k that does not have a p-th root in k. Then $f = x^p - a \in k[x]$ is irreducible by [Stacks, Tag 09HF]. In other words, $\sqrt[p]{a}$ is algebraic with minimal polynomial f. But the element is not separable algebraic over k since f has multiple roots in its splitting field $k(\sqrt[p]{a})$.
Hence we see, that the extension $k \subseteq k(\sqrt[p]{a})$ is not separably generated and thus, $k(\sqrt[p]{a})$ is not separable over k.

In the example it is shown that adjoining a p-th root is not separable. In fact, the next lemma states, that extensions of this type are the only way to generate non-separability:

Lemma 1.3.3. *A field k is perfect if and only if k has characteristic 0 or k is a field of characteristic $p > 0$ so that any element has a pth root in k.*

Proof. See [Stacks, Tag 030Z]. ∎

To establish a link between separable field extensions of k and *geometrically reduced* algebras over k, we may consider what happens if we extend the ground field of a reduced k-algebra by a separable extension.

Lemma 1.3.4. *If k is a field, S a reduced k-algebra and L/k a separable field extension, then $S \otimes_k L$ is reduced.*

Proof. The lemma is part of a more general statement: [Stacks, Tag 030U]. ∎

The assumption on L/k being separable is essential - the lemma will fail if we drop this:

Remark 1.3.5. Let k be a field of characteristic $p \neq 0$ and a an element of k, not having a p-th root in k. By Example 1.3.2 b), the minimal polynomial of $\sqrt[p]{a}$ is $x^p - a \in k[x]$ and the extension $k \subseteq k(\sqrt[p]{a})$ is not separable. Although $k(\sqrt[p]{a})$ is a reduced k-algebra, the algebra $k(\sqrt[p]{a}) \otimes_k k(\sqrt[p]{a})$ is not reduced:

$$\begin{aligned} k(\sqrt[p]{a}) \otimes_k k(\sqrt[p]{a}) &= k(\sqrt[p]{a}) \otimes_k k[x]/\langle x^p - a\rangle \\ &= k(\sqrt[p]{a})[x]/\langle x^p - a\rangle \\ &= k(\sqrt[p]{a})[x]/\langle\, (x - \sqrt[p]{a})^p \,\rangle \end{aligned}$$

And this has for example the nilpotent element $x - \sqrt[p]{a}$.

Definition 1.3.6. Let k be a field and S an k-algebra. Then S is called **geometrically reduced over** k if $S \otimes_k L$ is reduced for any field extension L of k.

As mentioned in the introduction of this section: for fields, this definition coincides with separability:

Proposition 1.3.7. *Let k be a field and L/k a field extension. Then L is geometrically reduced over k if and only if L is separable over k.*

Proof. We first prove the converse direction: Let K be an arbitrary extension field of k. Then K is reduced and since L/k is separable, we can apply 1.3.4 and obtain: $K \otimes_k L$ is reduced. Hence, L is geometrically reduced over k. For the other direction, we consider two cases:

- If $char(k) = 0$, then k is perfect by Lemma 1.3.3 and any field extension of k is separable, hence L/k is.
- If $char(k) = p \neq 0$, we get the equivalence by [Stacks, Tag 030W]

∎

The next lemma shows that over perfect fields, we do not have to deal with non-geometrically reduced algebras as long as we start with a reduced one. It also allows us to give easy examples.

Lemma 1.3.8. *Let k be a perfect field. Then any reduced k-algebra is geometrically reduced.*

Proof. Let S be a reduced k-algebra and L/k a field extension. Then L/k is separable since k is perfect and by Lemma 1.3.4, we obtain that $S \otimes_k L$ is reduced. ∎

As a consequence, we can show that over a perfect field k, radical ideals in an k-algebra stay radical when we extend the field.

Example 1.3.9. Let k be a perfect field, S an k-algebra, $I \trianglelefteq S$ a radical ideal and L/k a field extension. Then S/I is reduced and by Lemma 1.3.8, also $S/I \otimes_k L$ is reduced. Since field extensions are flat by Lemma A.2.3, we may apply A.2.1 and obtain that

$$S/I \otimes_k L = (S \otimes_k L)/I(S \otimes_k L)$$

is reduced. Hence, the extension of I to the L-algebra $S \otimes_k L$ is also a radical ideal.

1.4 Completion

The notion of completion can be transferred from fields to arbitrary modules over rings. In the field case, we use a topology and Cauchy sequences to define what the completion should be. Over modules, we will define *filtrations* - these naturally generate a topology which is compatible with the module structure. Instead of dealing with Cauchy sequences directly, we define the *completion of a module* to be a projective limit. This is again a module and it can be understood as set of all Cauchy sequences modulo a certain equivalence relation.
Furthermore, we will collect some basic properties of *completions* which are essential for later considerations in Section 2.4 and the "lift" of *invariants* to the *completion* of algebras in Chapter 4.

Definition 1.4.1. Let R be a ring and M an R-module.

a) A family of ideals $(I_n)_{n \in \mathbb{N}}$ in R forms a **filtration of** R if: $I_0 = R$, $I_{n+1} \subseteq I_n$ and $I_n I_m \subseteq I_{n+m}$.

b) A family of submodules $(M_n)_{n \in \mathbb{N}}$ of M forms a **filtration of** M if: $M_0 = M$, $M_{n+1} \subseteq M_n$ and $I_m M_n \subseteq M_{m+n}$.

Remark 1.4.2. If a ring R has a filtration $(I_n)_{n\in\mathbb{N}}$, then this induces a topology on R: consider the I_n as neighbourhoods of 0, then we obtain a topology, which is compatible with the ring structure - for details, we refer to the introduction of [Mat80, Chapter 23].
Similarly, we obtain a topology on an R-module M by using a filtration $(M_n)_{n\in\mathbb{N}}$. The resulting topology on M is compatible with the module structure and the underlying ring topology on R.

The following simple example is one of the most important filtrations used in commutative algebra - we will later focus on filtrations of this type.

Example 1.4.3. Let R be a ring, $\mathfrak{q}$ an ideal of R and M an R-module. The **$\mathfrak{q}$-adic filtration of** R is defined by $(\mathfrak{q}^n)_{n\in\mathbb{N}}$ and the **$\mathfrak{q}$-adic filtration of** M is given by the submodules $(\mathfrak{q}^n M)_{n\in\mathbb{N}}$. These obviously satisfy the conditions on filtrations.

When we are given a filtration $(M_n)_{n\in\mathbb{N}}$ on a module M, then the collection $\left\{M/M_n, i_n^{n+1}\right\}$, where $i_n^{n+1}: M/M_{n+1} \to M/M_n$ are the natural maps, is a projective system. Hence, we can consider the projective limit of the system, which is again a module.

Proposition 1.4.4. *Let R be a ring and M an R-module with filtration $(M_n)_{n\in\mathbb{N}}$. Then the projective limit $\varprojlim M/M_n$ is the topological completion of M: The set of all Cauchy sequences of elements of M modulo the equivalence relation: $(x_n) \sim (y_n)$ if for each $m \in \mathbb{N}$ there exists an m_0 so that the difference $x_n - y_n \in M_m$ for each $n \geq m_0$.*

Proof. See [AK70, Proposition 1.7]. ■

Definition 1.4.5. Let R be a ring and M an R-module with filtration $(M_n)_{n\in\mathbb{N}}$. The module $\varprojlim M/M_n$ is called **completion of** M and we denote it by $\widehat{M}^{M_n}$ or just $\widehat{M}$ if it is clear which filtration has been used.

Example 1.4.6. Let k be a field and $x_1, \ldots, x_n = \underline{x}$ indeterminates over k.

a) The completion of the polynomial ring $k[\underline{x}]$ with respect to the $\langle \underline{x} \rangle$-adic filtration is the formal power series ring $k[[\underline{x}]]$. For details, we refer to [Eis95, Chapter 7.1].

b) If k has a valuation, let $k\{\underline{x}\}$ denote the ring of power series, convergent with respect to the valuation on k. Then the $\langle \underline{x} \rangle$-adic completion of $k\{\underline{x}\}$ is $k[[\underline{x}]]$.

   Proof. If we factor out powers of the maximal ideal $\langle \underline{x} \rangle$, we "cut" power series at a certain degree. Hence, we have:

$$k\{\underline{x}\}/\langle \underline{x} \rangle^j = k[\underline{x}]/\langle \underline{x} \rangle^j$$

   Taking projective limits on both sides implies: $\widehat{k\{\underline{x}\}} = \widehat{k[\underline{x}]} = k[[\underline{x}]]$, by a). ■

c) Now we compute the completion of the localized polynomial ring $k[\underline{x}]_{\langle \underline{x} \rangle}$. The natural embedding $k[\underline{x}] \to k[[\underline{x}]]$ maps the set $k[\underline{x}] \setminus \langle \underline{x} \rangle$ into $k[[\underline{x}]]^*$. Hence, we get a map $\varphi : k[\underline{x}]_{\langle \underline{x} \rangle} \to k[[\underline{x}]]$ by the universal property of localization and φ satisfies: $\varphi(\langle \underline{x} \rangle^j_{\langle \underline{x} \rangle}) \subseteq \langle \underline{x} \rangle^j$. This induces the map

$$\begin{aligned} \overline{\varphi} : k[\underline{x}]_{\langle \underline{x} \rangle} / \langle \underline{x} \rangle^j_{\langle \underline{x} \rangle} &\longrightarrow k[[\langle \underline{x} \rangle]] / \langle \underline{x} \rangle^j \\ \overline{a/b} &\longmapsto \overline{a/b} \end{aligned}$$

- $\overline{\varphi}$ is injective: Let $\frac{a}{b} \in k[\underline{x}]_{\langle \underline{x} \rangle}$ and assume $\frac{a}{b} \in \langle \underline{x} \rangle^j$ in $k[[\underline{x}]]$. Then $a = b \cdot \frac{a}{b} \in \langle \underline{x} \rangle^j$. Since a is in $k[\underline{x}]$, we get that $\frac{a}{b} \in \langle \underline{x} \rangle^j_{\langle \underline{x} \rangle} \trianglelefteq k[\underline{x}]_{\langle \underline{x} \rangle}$.
- $\overline{\varphi}$ is surjective: For $\overline{f} \in k[[\underline{x}]]/\langle \underline{x} \rangle^j$, we can choose a representative $f \in k[[\underline{x}]]$ which has degree less than j. So f is a polynomial and therefore an element in $k[\underline{x}]_{\langle \underline{x} \rangle}$. The class of f in $k[\underline{x}]_{\langle \underline{x} \rangle} / \langle \underline{x} \rangle^j_{\langle \underline{x} \rangle}$ maps to the class of f in $k[[\underline{x}]]/\langle \underline{x} \rangle^j$.

Altogether, we get:

$$k[\underline{x}]_{\langle \underline{x} \rangle} / \langle \underline{x} \rangle^j_{\langle \underline{x} \rangle} = k[[\underline{x}]]/\langle \underline{x} \rangle^j = k[\underline{x}]/\langle \underline{x} \rangle^j$$

and thus: $\widehat{k[\underline{x}]_{\langle \underline{x} \rangle}} = k[[x]]$.

So we have three different rings but their completions coincide. We will use this fact later in Chapter 5 to justify why we can compute certain *invariants* over localized polynomial rings instead of convergent power series rings.

We will now focus on $\mathfrak{q}$-adic filtrations/completions and start collecting useful facts about these. The first remark concerns the lift of ideals under ring homomorphisms and the completion with respect to these lifted ideals.

Remark 1.4.7. If $\varphi : R \to S$ is a ring homomorphism and $\mathfrak{q}$ an ideal of R, then we can extend $\mathfrak{q}$ to an ideal of S by setting $\mathfrak{u} = \varphi(\mathfrak{q})S = \mathfrak{q}^e$. For a precise definition of the extension and contraction of ideals, we refer to A.1.1. Any S-module M is also an R-module and we have the equality:

$$\mathfrak{q}^n M = \varphi(\mathfrak{q}^n) M = \varphi(\mathfrak{q})^n M = \mathfrak{u}^n M$$

Hence, we get: $M/\mathfrak{q}^n M = M/\mathfrak{u}^n M$ and therefore the $\mathfrak{q}$-adic completion of M considered as R-module and the $\mathfrak{u}$-adic completion of M as S-module coincide:

$$\widehat{M}^{\mathfrak{q}} = \varprojlim M/\mathfrak{q}^n M = \varprojlim M/\mathfrak{u}^n M = \widehat{M}^{\mathfrak{u}}$$

Sometimes, different ideal filtrations induce the same topology and thus, have the same completion.

Lemma 1.4.8. *Let R be a ring and $\mathfrak{q}, \mathfrak{p}$ ideals of R. Suppose there exist natural numbers $c, d > 0$ so that: $\mathfrak{q}^c \subseteq \mathfrak{p}$ and $\mathfrak{p}^d \subseteq \mathfrak{q}$. Then for any R-module M, we have that $\mathfrak{q}$-adic completion coincides with $\mathfrak{p}$-adic completion:*

$$\widehat{M}^{\mathfrak{q}} = \widehat{M}^{\mathfrak{p}}$$

Proof. See [Stacks, Tag 0319] ■

Under suitable assumptions, completion is exact and it will turn out that it is even flat. This has an enormous influence on later results.

Proposition 1.4.9. *Let R be a Noetherian ring and $\mathfrak{q}$ an ideal in R. The functor $M \mapsto \widehat{M}$, which maps a finitely generated R-module to its $\mathfrak{q}$-adic completion is exact.*

Proof. This is [Mat80, Theorem 54]. ■

Corollary 1.4.10. *Let R be a Noetherian ring, $\mathfrak{q}$ an ideal of R and M a finitely generated R-module. If N is a submodule of M, then we have for the $\mathfrak{q}$-adic completion:*

$$\widehat{M/N} \cong \widehat{M}/\widehat{N}$$

Proof. The sequence $0 \to N \to M \to M/N \to 0$ is an exact sequence of finitely generated R-modules. Hence, we may apply Proposition 1.4.9 and obtain that the sequence $0 \to \widehat{N} \to \widehat{M} \to \widehat{M/N} \to 0$ is also exact. The claim follows then by the homomorphism theorem. ■

The $\mathfrak{q}$-adic completion of a ring R is, by definition, the projective limit of $R/\mathfrak{q}^n$. Hence, we can consider $\widehat{R}$ as submodule of $\prod_{n\in\mathbb{N}} R/q^n$ and similarly for an R-module M: $\widehat{M}$ is a submodule of $\prod_{n\in\mathbb{N}} M/\mathfrak{q}^n M$. We therefore get a canonical map $M \otimes_R \widehat{R} \to \widehat{M}$. The next theorem states, when this is actually an isomorphism:

Theorem 1.4.11. *For a Noetherian ring R, an ideal $\mathfrak{q}$ of R and a finitely generated R-module M, we have an isomorphism:*

$$M \otimes_R \widehat{R} \cong \widehat{M},$$

where $\widehat{\ }$ denotes the $\mathfrak{q}$-adic completion.

Proof. See [Mat80, Theorem 55]. ■

Corollary 1.4.12. *For a Noetherian ring R, an ideal $\mathfrak{q}$ and finitely generated R-modules $M_1, \ldots, M_n$, the $\mathfrak{q}$-adic completion commutes with the direct sum of the M_i:*

$$\left(\bigoplus_{i=1}^{n} M_i\right)^{\widehat{\ }} \cong \bigoplus_{i=1}^{n} \widehat{M_i}$$

Proof. The direct sum of the M_i is finitely generated, hence we may apply Theorem 1.4.11 and use that the tensor product commutes with direct sums. ■

Corollary 1.4.13. *For a Noetherian ring R and an ideal $\mathfrak{q}$ of R we have that the $\mathfrak{q}$-adic completion $\widehat{R}$ is flat over R.*

Proof. This is a direct consequence of Proposition 1.4.9 and Theorem 1.4.11. ■

The next Proposition shows that ideals behave well with respect to completion. This allows us to give some examples.

Proposition 1.4.14. *Let R be a Noetherian ring, I and $\mathfrak{q}$ ideals of R and M a finitely generated R-module. If R and M are filtered with a $\mathfrak{q}$-adic filtration, then:*

a) $I\,\widehat{M} = \widehat{(IM)} = \widehat{I}\,\widehat{M}$

b) $\widehat{M}/I\,\widehat{M} = (\widehat{M/IM})$

In particular: $M/\mathfrak{q}^n M = \widehat{M}/\mathfrak{q}^n\widehat{M} = \widehat{M}/\hat{\mathfrak{q}}^n\widehat{M}$.

Proof. This is part of [AK70, Proposition 1.19]. ■

Example 1.4.15. Let k denote a field and $x_1, \ldots, x_n = \underline{x}$ indeterminates over k. Set $R = k[\underline{x}]/I$, then $\widehat{R} = k[[\underline{x}]]/Ik[[\underline{x}]]$ by Proposition 1.4.14 and Example 1.4.6. We see, that the completion of a finitely generated k-algebra is a *formal analytic algebra* over k.

If we consider the $\mathfrak{q}$-adic completion of a module M as submodule of the product of the $M/q^n M$, like above, we always have a canonical map $M \to \widehat{M}$ with kernel $\bigcap q^n M$. Intuitively, the module should embed into its completion - in this case, the induced topology on M is called **Hausdorff**. We can show that for finitely generated modules over local Noetherian rings, the topology induced by the maximal ideal is Hausdorff.

Theorem 1.4.16. *Let R be a Noetherian ring, $\mathfrak{q} \trianglelefteq R$ an ideal and M a finitely generated R-module. The kernel $\bigcap \mathfrak{q}^n M$ of the canonical map $M \to \widehat{M}$ consists of all elements of M that get annihilated by some element of $1 + \mathfrak{q}$.*

Proof. See [AM69, Theorem 10.17]. ■

Corollary 1.4.17. *Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module. Then the topology induced by $\mathfrak{m}$ on M is Hausdorff: $\bigcap \mathfrak{m}^n M = 0$. In particular: $M \to \widehat{M}$ is injective.*

Proof. By Theorem 1.4.16, we know that $\bigcap \mathfrak{m}^n M$ consists of all elements annihilated by some element from $1 + \mathfrak{m}$. But since R is local, the elements in $1 + \mathfrak{m}$ are all units and these only annihilate 0. Hence, $\bigcap \mathfrak{m}^n M = 0$. ∎

When completing a ring, we may ask which properties of the rings survive the completion. As one expects, the Noetherian property seems to persist anything:

Proposition 1.4.18. *If R is a Noetherian ring and $\mathfrak{q} \trianglelefteq R$ an ideal, then $\widehat{R} = \varprojlim R/\mathfrak{q}^n$ is also Noetherian.*

Proof. This is [AK70, Proposition 1.22]. ∎

The property of being local also transfers to the completion:

Proposition 1.4.19. *Let R be a ring and $\mathfrak{q}$ an ideal of R. Then the map*

$$\begin{aligned} \{\mathfrak{m} \trianglelefteq R \,|\, \mathfrak{m} \textit{ maximal}, \ \mathfrak{q} \subseteq \mathfrak{m}\} &\longrightarrow \left\{\mathfrak{n} \trianglelefteq \widehat{R} \,\middle|\, \mathfrak{n} \textit{ maximal}\right\} \\ \mathfrak{m} &\longmapsto \widehat{\mathfrak{m}} \end{aligned}$$

is a bijection. In particular, if R is local with maximal ideal $\mathfrak{m}$, then $\widehat{R}$ is local with maximal ideal $\widehat{\mathfrak{m}}$.

Proof. See [AK70, Proposition 1.24]. ∎

As an application, we can deduce that the dimension does not change if we pass to the completion:

Proposition 1.4.20. *Let R be a local ring and let $\mathfrak{q}$ be an ideal of R. Then we have for the $\mathfrak{q}$-adic completion:* $\dim \widehat{R} = \dim R$.

Proof. Since the map $R \to \widehat{R}$ is a flat local homomorphism by Proposition 1.4.19 and Corollary 1.4.13, we get by Proposition A.2.5: $\dim \widehat{R} = \dim R + \dim \widehat{R}/\mathfrak{m}\widehat{R}$. But since $\mathfrak{m}\widehat{R}$ is the maximal ideal of $\widehat{R}$, $\widehat{R}/\mathfrak{m}\widehat{R}$ is a field and therefore $\dim \widehat{R}/\mathfrak{m}\widehat{R} = 0$. ∎

2. Normalization

The following chapter mainly treats the integral closure of ring extensions and in particular the normalization of a ring and its behaviour with respect to completion and field extension. Starting with basic definitions and properties around integrality in 2.1, the chapter continues with giving criteria, when the normalization is actually finite - see 2.2 . In Section 2.3, normal rings are introduced and it is shown that under suitable assumptions, the normalization is actually normal. The last two sections, 2.4 and 2.5 concern the commutativity between normalization and completion, respectively field extension.

2.1 The integral closure and integrally closed rings

Like algebraic elements in a field extension, *integral* elements are elements in a ring extension $R \subseteq S$, satisfying certain polynomial relations. The set of all these elements is called *integral closure of R in S* and is, in fact, a new over ring of R. If S is the total ring fractions, the integral closure of R in S is also called *normalization* of R. A goal of this section is to state the result that the *normalization* of a reduced ring factors into *normalizations* of integral domains - this is useful when computing *normalizations*, as one can see in Example 2.1.16. In the subsequent section, 2.2, we will state some results, when the *normalization* is actually a finitely generated module.

Definition 2.1.1. Let $R \subseteq S$ be a ring extension and $\alpha \in S$.

a) The element α is called **integral over** R if there is a monic polynomial $f \in R[x]$ so that α is a zero of f.

b) The ring S is called **integral over** R if any element of S is integral over R.

c) The **integral closure of** R **in** S is the set of all elements of S that are integral over R, denoted by: $\mathrm{Int}_S(R)$.

One major difference between algebraic and integral elements is, that in the case of a ring extension, we cannot always guarantee to have a monic polynomial. The following example shows, that it is possible for an element to satisfy a polynomial relation without being integral. For algebraic elements, this cannot happen: if an element is the zero of a polynomial over a field, then we can always multiply by a unit to make the polynomial monic.

Example 2.1.2. Let R be a unique factorization domain (UFD) and $Q(R)$ its field of fractions. Then:

$$\mathrm{Int}_{Q(R)}(R) = R$$

A proof is given in [HS06, Proposition 2.1.5]. As an immediate consequence, we obtain that the fraction $\frac{1}{2} \in \mathbb{Q}$ is not integral over $\mathbb{Z}$ although it is a zero of the polynomial $2x - 1 \in \mathbb{Z}[x]$.

The following lemma describes the relation between finitely generated R-modules and integral elements.

Lemma 2.1.3. *If $R \subseteq S$ is a ring extension and $x_1, \ldots, x_n \in S$, then the following statements are equivalent:*

a) The elements $x_1, \ldots, x_n$ are integral over R.

b) The module $R[x_1, \ldots, x_n]$ is finitely generated over R.

c) There is a finitely generated R-module $M \subseteq S$ that is also a faithful $R[x_1, \ldots, x_n]$-module.

Proof. See [HS06, Lemma 2.1.9] ■

With the help of this, one can easily show that the integral closure of a ring R in a ring S is again a ring. Equivalently one could say: the sum and the product of integral elements are again integral.

Corollary 2.1.4. *Let $R \subseteq S$ be a ring extension. Then $\mathrm{Int}_S(R)$ is also a ring.*

Proof. Let $x, y \in \mathrm{Int}_S(R)$, then by Lemma 2.1.3 we get that $R[x, y]$ is a finitely generated R-module. The elements $x + y$, xy and $-x$ are in $R[x, y]$. So $R[x, y]$ is a faithful module over $R[x + y]$, $R[xy]$ and $R[-x]$. Applying Lemma 2.1.3 with $M = R[x, y]$, we obtain that $x + y$, xy and $-x$ are again integral over R. ■

A further corollary of the lemma above shows that integral extensions are in particular transitive.

Corollary 2.1.5. *Let $R \subseteq S \subseteq T$ be an extension of rings. Then T is integral over R if and only if T is integral over S and S is integral over R.*

Proof. This is [HS06, Corollary 2.1.12] ■

Now assume, we have an integral ring extension $R \subseteq S$ and we extend scalars by an R-algebra T. We get the following diagram:

$$\begin{tikzcd} S \arrow[r] & S \otimes_R T \\ R \arrow[u, "int"] \arrow[r] & R \otimes_R T = T \arrow[u] \end{tikzcd}$$

The next lemma answers the question if the T-algebra $S \otimes_R T$ is integral over T.

Lemma 2.1.6. *If $R \subseteq S$ is a ring extension and S is integral over R, then for any R-algebra T, we have: $S \otimes_R T$ is integral over $T = R \otimes_R T$.*

Proof. Let $x \otimes t \in S \otimes_R T$, then $x \in S$ is integral over R. Hence, we get a relation:

$$0 = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

with $a_i \in R$. Now we tensorize by t^n and make use of bilinearity and multiplication on $S \otimes_R T$:

$$\begin{aligned} 0 &= (x^n + a_{n-1}x^{n-1} + \cdots + a_0) \otimes t^n \\ &= x^n \otimes t^n + a_{n-1}x^{n-1} \otimes t^n + \cdots + a_0 \otimes t^n \\ &= (x \otimes t)^n + (a_{n-1} \otimes t)(x \otimes t)^{n-1} + \cdots + a_0 \otimes t^n \end{aligned}$$

We constructed a monic relation for $x \otimes t$ with coefficients in $R \otimes_R T = T$, so $x \otimes t$ is integral over T. For an arbitrary element $y = \sum x_i \otimes t_i$ in $S \otimes_R T$, we get that any summand is integral over T. Since the integral closure $\mathrm{Int}_{S \otimes_R T}(T)$ is a ring by Corollary 2.1.4, we obtain that y is integral over T. ■

If $R \subseteq S$ is a ring extension, then one would expect that all elements of S which are integral over $\mathrm{Int}_S(R)$ lie already in the integral closure. In fact, this is true:

Lemma 2.1.7. *Let $R \subseteq S$ be a ring extension. Then:* $\mathrm{Int}_S(\mathrm{Int}_S(R)) = \mathrm{Int}_S(R)$

Proof. The inclusion "$\supseteq$" is clear. For the converse inclusion note that $\mathrm{Int}_S(\mathrm{Int}_S(R))$ is integral over $\mathrm{Int}_S(R)$, which is itself integral over R. Hence $\mathrm{Int}_S(\mathrm{Int}_S(R))$ is integral over R by 2.1.5 and thus a subset of $\mathrm{Int}_S(R)$. ■

The integral closure is also compatible with localization:

Lemma 2.1.8. *For a ring extension $R \subseteq S$ and a multiplicatively closed set $W \subseteq R$, we have:*

$$W^{-1}\,\mathrm{Int}_S(R) = \mathrm{Int}_{W^{-1}S}(W^{-1}R)$$

Proof. Let $\frac{x}{w}$ be from the left hand side, where $x \in S$ is integral over R and $w \in W$. The element x satisfies a polynomial relation

$$x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 = 0$$

with $r_j \in R$. Multiplying with $\frac{1}{w^n}$, we obtain another polynomial relation with coefficients in $W^{-1}R$:

$$\left(\frac{x}{w}\right)^n + \frac{r_{n-1}}{w} \cdot \left(\frac{x}{w}\right)^{n-1} + \cdots + \frac{r_1}{w^{n-1}} \cdot \left(\frac{x}{w}\right) + \frac{r_0}{w^n} = 0$$

This shows that $\frac{x}{w} \in W^{-1}S$ is integral over $W^{-1}R$.

For the converse direction, let $\frac{x}{w} \in \operatorname{Int}_{W^{-1}S}(W^{-1}R)$. Then $\frac{x}{w}$ is integral over $W^{-1}R$, so it satisfies a relation

$$\left(\frac{x}{w}\right)^n + \frac{r_{n-1}}{w_{n-1}} \cdot \left(\frac{x}{w}\right)^{n-1} + \cdots + \frac{r_1}{w_1} \cdot \left(\frac{x}{w}\right) + \frac{r_0}{w_0} = 0$$

with $r_j \in R$, $w_j \in W$. Set $\tilde{w} = \prod_{j=0}^{n-1} w_j \in W$ and multiply the relation with w^n and $\tilde{w}^n$:

$$(\tilde{w}x)^n + \frac{r_{n-1}w\tilde{w}}{w_{n-1}} \cdot (\tilde{w}x)^{n-1} + \cdots + \frac{r_1 w^{n-1}\tilde{w}^{n-1}}{w_1} \cdot (\tilde{w}x) + \frac{r_0 w^n \tilde{w}^n}{w_0} = 0$$

This has coefficients in R since $w_{n-1}|\tilde{w}$. So we get that $\tilde{w}x \in S$ is integral over R and thus:

$$\frac{x}{w} = \frac{\tilde{w}x}{\tilde{w}w} \in W^{-1}\operatorname{Int}_S(R)$$

■

If we start with an integral extension of integral domains $R \subseteq S$, then we can relate the maximal primes of S to the maximal primes of R:

Proposition 2.1.9. *Let $R \subseteq S$ be an integral extension of integral domains. If Q is a prime ideal of S, then Q is a maximal ideal of S if and only if $Q \cap R$ is a maximal ideal of R.*

Proof. See [Eis95, Corollary 4.17]. ■

The following result is called *Incomparability*. In fact, this statement is one of the four most basic theorems that treat the behaviour of prime ideals under integral extensions. The other results are called *Going-Up*, *Going-Down* and *Lying-Over* and we refer to [HS06, Section 2.2] for the statements.

Theorem 2.1.10. *Let $R \subseteq S$ be an integral ring extension and $P \subseteq Q$ prime ideals of S. If $P \cap R = Q \cap R$, then $P = Q$.*

Proof. See [HS06, Theorem 2.2.3]. $\blacksquare$

We will now consider *normalizations*: integral closures of rings in their total ring of fractions. As we have already seen in Example 2.1.2, there are rings whose integral elements in the total ring of fractions lie already in the ring. Such rings are called *integrally closed.*

Definition 2.1.11. Let R be a reduced ring, then its total ring of fractions $Q(R)$ is an over ring. The integral closure of R in $Q(R)$ is called **integral closure of** R or **normalization of** R and it is denoted by: $\overline{R}$. The ring R is said to be **integrally closed** if: $\overline{R} = R$.

In fact, we have already seen integrally closed rings:

Remark 2.1.12. Let R be a UFD, then Example 2.1.2 shows that $\mathrm{Int}_{Q(R)}(R) = R$. This means R is integrally closed.

Proposition 2.1.13. *Let $R \subseteq S$ be an integral ring extension, where R is reduced and $S \subseteq Q(R)$. Then S is integrally closed if and only if $S = \overline{R}$.*

Proof. First of all, S is also a reduced ring by Lemma 1.2.6, so the claim above makes sense without the assumption on S being reduced.
If S is integrally closed, we have to show the set equality. Let $x \in S$, then x is integral over R and $x \in Q(R)$. Thus, x lies in $\mathrm{Int}_{Q(R)}(R) = \overline{R}$. We get a chain of inclusions:

$$R \subseteq S \subseteq \overline{R} \subseteq Q(R) \subseteq Q(S)$$

and since we know that $\overline{R}$ is integral over R, we get by Corollary 2.1.5 that $\overline{R}$ is integral over S. Hence, $\overline{R} \subseteq \mathrm{Int}_{Q(S)}(S) = S$.
For the converse direction, we immediately get by Lemma 2.1.7 that the normalization of R and thus S, is integrally closed. $\blacksquare$

For the next proposition we need the fact, that the normalization of an integral domain R is again an integral domain. This is clearly true, since $\overline{R}$ is a subring of the field $Q(R)$.

Proposition 2.1.14. *Let R be an integral domain and let $\mathfrak{J}(R)$ and $\mathfrak{J}(\overline{R})$ denote the Jacobson radicals of R and $\overline{R}$. Then the natural inclusion $R \to \overline{R}$ maps $\mathfrak{J}(R)$ into $\mathfrak{J}(\overline{R})$.*

Proof. Let $\mathfrak{n}$ be a maximal ideal of $\overline{R}$. Since $R \subseteq \overline{R}$ is an integral extension of integral domains, we may apply Proposition 2.1.9 and obtain that $\mathfrak{n}^c = \mathfrak{n} \cap R$ is a maximal ideal of R. Therefore, $\mathfrak{n}^c \supseteq \mathfrak{J}(R)$ and by Remark A.1.2, we have:

$$\mathfrak{J}(R)\overline{R} = \mathfrak{J}(R)^e \subseteq \mathfrak{n}^{ce} \subseteq \mathfrak{n}$$

Hence, $\mathfrak{J}(R)\overline{R} \subseteq \mathfrak{J}(\overline{R})$ $\blacksquare$

In Lemma 1.2.11 we saw, that for a reduced ring R, the total ring of fractions is a product of fields. We obtain a similar statement for the normalization: $\overline{R}$ can be decomposed into a product of normalizations $\overline{R/P}$, where P runs through the minimal primes of R. Geometrically this means, that the normalization of a reduced variety is the product of the normalizations of the branches.

Proposition 2.1.15. *Let R be a reduced ring and $P_1, \dots, P_r$ its minimal primes. Then:*

$$\overline{R} = \overline{R/P_1} \times \cdots \times \overline{R/P_r},$$

where $\overline{R/P_i}$ is the normalization of R/P_i.

Proof. A proof can be found in [HS06, Corollary 2.1.13]. ■

Example 2.1.16. Let k be a field.

a) Set $R = k[x,y]/\langle xy \rangle$. We want to compute the normalization of R: In Example 1.2.12, we saw that R modulo its two minimal primes gives us the polynomial rings: $k[y]$ and $k[x]$. These are UFDs, hence they are integrally closed by Remark 2.1.12. Now we apply Proposition 2.1.15 and get:

$$\overline{R} = k[y] \times k[x]$$

b) Let now R denote the reduced ring $k[x,y]/\langle y^4 - x^6 \rangle$. Then we can decompose the polynomial into two irreducible factors: $y^4 - x^6 = (y^2 - x^3) \cdot (y^2 + x^3)$. Hence, the minimal primes are $\langle y^2 - x^3 \rangle$ and $\langle y^2 + x^3 \rangle$. By Proposition 2.1.15, we can deduce:

$$\overline{R} = \overline{k[x,y]/\langle y^2 - x^3 \rangle} \times \overline{k[x,y]/\langle y^2 + x^3 \rangle}$$

Let us first consider the normalization of $k[x,y]/\langle y^2 - x^3 \rangle$. Therefore, let φ be the map:

$$\begin{aligned} k[x,y] &\longrightarrow k[t^2, t^3] \subseteq k[t] \\ x &\longmapsto t^2 \\ y &\longmapsto t^3 \end{aligned}$$

Then φ is surjective and $\mathrm{Ker}(\varphi) = \langle y^2 - x^3 \rangle$. We get an isomorphism $k[x,y]/\langle y^2 - x^3 \rangle \cong k[t^2, t^3]$. Furthermore t is a root of the monic polynomial $T^2 - t^2 \in k[t^2, t^3][T]$ and hence, integral over $k[t^2, t^3]$. Since sums and products of integral elements are again integral by Lemma 2.1.3, we get that $k[t]$ is integral over $k[t^2, t^3]$. The polynomial ring $k[t]$ is a UFD and therefore integrally closed, see Remark 2.1.12. We can also deduce that $Q(k[t]) = k(t) = Q(k[t^2, t^3])$ and altogether we are in the situation of Proposition 2.1.13. Hence, $k[t]$ is the normalization of

$k[t^2, t^3]$.
A similar argumentation shows that $k[x,y]/\langle y^2+x^3\rangle \cong k[z^2,z^3]$ and the normalization of $k[z^2,z^3]$ is $k[z]$.
Finally, we obtain:
$$\overline{R} \cong k[t] \times k[z]$$
Geometrically, the curve y^4-x^6 consists of two cusps: y^2-x^3 and y^2+x^3. The normalization, that we were able to construct with Proposition 2.1.15, is the product of two lines. So the normalization of the curve is the product of the normalization of its branches.

Another useful property is a consequence of the facts, that the ring of total fractions and the integral closure commute with localization if we consider a reduced ring.

Proposition 2.1.17. *Let R be a reduced ring with finitely many minimal primes and W a multiplicatively closed subset of R. Then we have:*

$$\overline{W^{-1}R} = W^{-1}\overline{R}$$

Proof. This is an application of Lemma 1.2.13 and Lemma 2.1.8:

$$\begin{aligned}\overline{W^{-1}R} &= \mathrm{Int}_{Q(W^{-1}R)}(W^{-1}R)\\ &= \mathrm{Int}_{W^{-1}Q(R)}(W^{-1}R)\\ &= W^{-1}\,\mathrm{Int}_{Q(R)}(R)\\ &= W^{-1}\overline{R}\end{aligned}$$ ■

2.2 Finitely generated normalizations

The main goal of this section is to show, that the normalization of an algebra, which is reduced essentially of finite type, is again an algebra of the same kind. We first handle finitely generated algebras over fields. Therefore, we will use a powerful theorem, proven by Emmy Noether, which gives us a first hint, when an algebra is *normalization-finite.* After that we "localize" our results.

Definition 2.2.1. We call a reduced ring R **normalization-finite** if $\overline{R}$ is a finitely generated module over R.

Example 2.2.2. Let k be a field and $R = k[x,y]/\langle x,y\rangle$. Then we already know by Example 2.1.16, that the normalization of R is $k[y] \times k[x]$ and the minimal primes are $\overline{\langle x\rangle}$ and $\overline{\langle y\rangle}$. The polynomial ring $k[y]$ is generated by 1 as $R/\overline{\langle x\rangle}$-module, hence it also generated by 1 as R-module. Similarly, 1 generates $k[x]$ as R-module and thus: $k[y] \times k[x]$ is generated by $(1,0)$ and $(0,1)$ as R-module. Altogether: R is normalization-finite.

To reach our goal we start with finitely generated algebras over fields and the aforementioned theorem of Emmy Noether:

Theorem 2.2.3. *Let k be a field and R a finitely generated k-algebra which is a domain. If $L/Q(R)$ is a finite field extension, then* $\operatorname{Int}_L(R)$ *is a finitely generated R-module.*
In particular: R is normalization-finite.

Proof. See [Eis95, Proposition 4.14]. ■

Before we generalize the result to reduced algebras, we mention a short fact that we will use a few times in our considerations.

Remark 2.2.4. Let k be a field, R a finitely generated k-algebra and S a ring, which is a finitely generated R-module, then S is a finitely generated k-algebra.

Proof. We get generators of S as an R-module: $s_1, \dots, s_l \in S$, hence the map $R[\underline{y}] = R[y_1, \dots, y_l] \to S$, which sends y_i to s_i, is surjective. We also have an surjective map $k[\underline{x}] \to R$ since R is finitely generated as k-algebra. If we tensor this with $k[\underline{y}]$ over k, we obtain a sequence of surjective maps:

$$k[\underline{x}, \underline{y}] = k[\underline{x}] \otimes_k k[\underline{y}] \longrightarrow R \otimes_k k[\underline{y}] = R[\underline{y}] \longrightarrow S$$

Hence, also S is a finitely generated k-algebra. ■

Corollary 2.2.5. *Let k be a field and R a reduced finitely generated k-algebra. Then $\overline{R}$ is a reduced finitely generated k-algebra and R is normalization-finite.*

Proof. The ring R is Noetherian, hence the set $\operatorname{Min}(R)$ is finite by Lemma 1.1.5. We can therefore apply Proposition 2.1.15 and obtain:

$$\overline{R} = \prod_{P \in \operatorname{Min}(R)} \overline{R/P}$$

The integral domains R/P are finitely generated R-modules and hence, they are also finitely generated k-algebras by Remark 2.2.4. Applying Theorem 2.2.3, we get that $\overline{R/P}$ are finitely generated as R/P-modules. But then the $\overline{R/P}$ are also finitely generated as R-modules and their product $\overline{R}$ also has finitely many generators over R. Finally, Remark 2.2.4 and Lemma 1.2.6 imply that $\overline{R}$ is a reduced finitely generated k-algebra. ■

Now we pass to localizations of finitely generated algebras and show that the result is still valid:

Proposition 2.2.6. *If k is a field and S is reduced essentially of finite type over k. Then $\overline{S}$ is reduced essentially of finite type over k and S is normalization-finite.*

Proof. We can write $S = W^{-1}R$, where R is a finitely generated k-algebra and W a multiplicatively closed set in R. Since S is reduced and the nilradical commutes with localization by Lemma 1.2.3, we can deduce:

$$S = S/\mathfrak{N}(S) = W^{-1}R/W^{-1}\mathfrak{N}(R) = W^{-1}(R/\mathfrak{N}(R))$$

The ring $R/\mathfrak{N}(R)$ is still a finitely generated k-algebra, hence we may assume that R is reduced.
By Corollary 2.2.5, $\overline{R}$ is a finitely generated k-algebra and R is normalization-finite. If we apply Proposition 2.1.17, then we get:

$$\overline{S} = \overline{W^{-1}R} = W^{-1}\overline{R}$$

Thus, $\overline{S}$ is a localization of a finitely generated k-algebra and therefore essentially of finite type over k and it is reduced by Lemma 1.2.6.
Since $\overline{R}$ is a finitely generated R-module, also $\overline{S} = W^{-1}\overline{R}$ is finitely generated over $W^{-1}R = S$. ∎

We did not mention yet, if the normalization of a Noetherian ring is again Noetherian. In the case that our ring is normalization-finite, this is actually true:

Lemma 2.2.7. *If the reduced Noetherian ring R is normalization-finite, then $\overline{R}$ is a Noetherian ring.*

Proof. The normalization is finitely generated as an R-module. Hence, $\overline{R}$ is Noetherian as module over R. But any chain of ideals in $\overline{R}$ is a chain of R-submodules and such sequences stabilize. ∎

Starting with a normalization-finite local ring, we may lose locality when we pass to the normalization. But we cannot have infinitely many maximal prime ideals occurring:

Proposition 2.2.8. *Let R be a reduced local normalization-finite ring with finitely many minimal primes. Then the normalization $\overline{R}$ is semi-local.*

Proof. First, we may assume that R is an integral domain and we denote by $\mathfrak{m}$ the maximal ideal of R. Since $\overline{R}$ is a finitely generated R-module, $\overline{R}/\mathfrak{m}\overline{R}$ is a finitely generated vector space over the field $R/\mathfrak{m}$. Hence, $\overline{R}/\mathfrak{m}\overline{R}$ is finite dimensional and therefore Artinian. By Proposition A.3.1, we get that $\overline{R}/\mathfrak{m}\overline{R}$ has finitely many maximal ideals. The ring R is an integral domain, so we get that $\mathfrak{m}\overline{R} \subseteq \mathfrak{J}(\overline{R})$ by Proposition 2.1.14. Hence, the maximal ideals of $\overline{R}$ and the maximal ideals of $\overline{R}/\mathfrak{m}\overline{R}$ are in one-to-one correspondence and therefore, $\overline{R}$ has finitely many maximal ideals.
Now let R be reduced with minimal primes $P_1, \ldots, P_r$. Then we know from Proposition 2.1.15:

$$\overline{R} = \prod_{i=1}^{r} \overline{R/P_i}$$

The first considered case applies to the local rings R/P_i and therefore any $\overline{R/P_i}$ is semi-local. This implies, that $\overline{R}$ is semi-local, since any maximal ideal $\mathfrak{n}$ of $\overline{R}$ is of the form:

$$\mathfrak{n} = \overline{R/P_1} \times \cdots \times \mathfrak{m}_i \times \cdots \times \overline{R/P_r},$$

where $\mathfrak{m}_i$ is a maximal ideal of $\overline{R/P_j}$. ■

As a last result in this section, we connect normalization-finite local rings with completion: When we consider modules over the semi-local normalization, we may form the completion with respect to the Jacobson radical. The next result states that this is the same as considering the module over the ground ring and forming the completion with respect to the maximal ideal.

Proposition 2.2.9. *Let R be a reduced local normalization-finite Noetherian ring with maximal ideal $\mathfrak{m}$. Then:*

a) Under the natural map $R \to \overline{R}$, $\mathfrak{m}$ maps into $\mathfrak{J}(\overline{R})$

b) $\sqrt{\mathfrak{m}^e} = \mathfrak{J}(\overline{R})$

c) There exists a $t \in \mathbb{N}$, $t > 0$ so that $\mathfrak{J}(\overline{R})^t \subseteq \mathfrak{m}^e$

d) For any $\overline{R}$-module, the $\mathfrak{J}(\overline{R})$-adic completion and the $\mathfrak{m}$-adic completion coincide.

Proof. The natural map $R \to \overline{R}$ factors:

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, "\Delta"'] & & \overline{R} = \prod_{P\in\mathrm{Min}(R)} \overline{R/P} \\ & \prod_{P\in\mathrm{Min}(R)} R/P \arrow[ur] & \end{tikzcd}$$

The map Δ sends r to $(\overline{r}, \ldots, \overline{r})$ and is injective:

$$\mathrm{Ker}(\Delta) = \bigcap_{P\in\mathrm{Min}(R)} P = \mathfrak{N}(R) = 0$$

Hence, the ideal $\mathfrak{m}$ maps into the ideal $\prod_{P\in\mathrm{Min}(R)} \mathfrak{m}/P$. Since the $\mathfrak{m}/P$ are the maximal ideals of R/P, we obtain that $\mathfrak{m}/P = \mathfrak{J}(R/P)$ and

$$\mathfrak{m} \hookrightarrow \prod_{P\in\mathrm{Min}(R)} \mathfrak{J}(R/P)$$

By Proposition 2.1.14, we obtain: $\mathfrak{J}(R/P) \hookrightarrow \mathfrak{J}(\overline{R/P})$ and if we use Proposition 2.1.15 and inductively apply Lemma A.3.2, we get:

$$\mathfrak{m} \hookrightarrow \prod_{P \in \operatorname{Min}(R)} \mathfrak{J}(\overline{R/P}) = \mathfrak{J}\left(\prod_{P \in \operatorname{Min}(R)} \overline{R/P}\right) = \mathfrak{J}(\overline{R})$$

This proves part a).

Now that we have proven that $\mathfrak{m}$ maps into $\mathfrak{J}(\overline{R})$, we get that $\mathfrak{m}^e$ is contained in $\mathfrak{J}(\overline{R})$. Since the Jacobson radical is a intersection of prime ideals, we can deduce:

$$\sqrt{\mathfrak{m}^e} \subseteq \sqrt{\mathfrak{J}(\overline{R})} = \mathfrak{J}(\overline{R})$$

From commutative algebra, we know that we can write:

$$\sqrt{\mathfrak{m}^e} = \bigcap_{\substack{P \in \operatorname{Spec}(\overline{R}) \\ P \supseteq \mathfrak{m}^e}} P$$

Now let P be such a prime ideal of $\overline{R}$, containing $\mathfrak{m}^e$. Then $P^c \supseteq \mathfrak{m}^{ec} \supseteq \mathfrak{m}$ by Lemma A.1.2. Since P^c is prime in R and $\mathfrak{m}$ maximal, we have: $P^c = \mathfrak{m}$. There exists a maximal ideal $\mathfrak{n}$ of $\overline{R}$ that contains P. Since $\mathfrak{n}$ also contains $\mathfrak{m}^e$, we get with the same argument: $P^c = \mathfrak{n}^c$. Hence by Incomparability, Theorem 2.1.10, we can deduce that $P = \mathfrak{n}$. Therefore, we obtain:

$$\bigcap_{\substack{P \in \operatorname{Spec}(\overline{R}) \\ P \supseteq \mathfrak{m}^e}} P \supseteq \bigcap_{\mathfrak{n} \text{ maximal in } \overline{R}} \mathfrak{n} = \mathfrak{J}(\overline{R})$$

Thus, we have proven part b).

To prove part c), we first note that $\overline{R}$ is Noetherian by 2.2.7. Then $\overline{R}/\mathfrak{m}^e$ is also Noetherian and by Remark 1.2.4, the ideal

$$\mathfrak{N}(\overline{R}/\mathfrak{m}^e) = \sqrt{\mathfrak{m}^e}/\mathfrak{m}^e = \mathfrak{J}(\overline{R})/\mathfrak{m}^e$$

is nilpotent. This means, there exists an $t \in \mathbb{N}$, $t > 0$ so that

$$(\mathfrak{J}(\overline{R})^t + \mathfrak{m}^e)/\mathfrak{m}^e = \overline{0}$$

Hence, $\mathfrak{J}(\overline{R})^t \subseteq \mathfrak{m}^e$.

Part d) is a direct consequence of what we have shown so far, Lemma 1.4.8 and Remark 1.4.7. ■

2.3 Normal rings

Normal rings are a generalization of integrally closed rings. In fact, we will see that the two definitions coincide under suitable assumptions and we will give a criterion, when the normalization is actually a *normal ring.* We will need the notion of *normal rings* in Section 2.5 to lift the normalization of an algebra when extending the ground field.

Definition 2.3.1. An arbitrary ring R is said to be **normal** if R_P is an integrally closed integral domain for every prime ideal $P \in \operatorname{Spec} R$.

We can immediately deduce from the definition that a normal ring is reduced. Locally, it is an integral domain, so we can apply Lemma 1.2.10 and obtain that the ring is reduced.
As mentioned before, normal rings are a generalization of integrally closed rings:

Lemma 2.3.2. *A normal ring is reduced and integrally closed.*

Proof. This is [Stacks, Tag 034M]. ■

If we know that we deal with a reduced ring, having finitely many minimal prime ideals, then the two notions are equivalent:

Proposition 2.3.3. *Let R be a reduced ring with finitely many minimal primes. Then R is normal if and only if R is integrally closed.*

Proof. If R is normal, then this is Lemma 2.3.2.

For the converse implication, assume that R is integrally closed and let $P \in \operatorname{Spec}(R)$. Then, we know that the total ring of fractions commutes with localization by Lemma 1.2.13:

$$R_P = \operatorname{Int}_{Q(R)}(R)_P = \operatorname{Int}_{Q(R)_P}(R_P) = \operatorname{Int}_{Q(R_P)}(R_P)$$

So R_P is integrally closed and we just have to show that it is also an integral domain. The ring R_P is reduced by Lemma 1.2.10 and has finitely many minimal primes by the ideal correspondence between R_P and R. Hence, we can apply Proposition 2.1.15 and Lemma 1.2.11 to get:

$$R_P = \overline{R_P} = \prod_{A \in \operatorname{Min}(R_P)} \overline{R_P/A} \text{ and } Q(R_P) = \prod_{A \in \operatorname{Min}(R_P)} Q(R_P/A)$$

Since R_P is local, it cannot be a product of rings, unless $r = 1$, by Lemma A.3.3. Thus, $Q(R_P) = Q(R_P/A_i)$ is a field and R_P an integral domain. ■

Since by Lemma 1.1.5, any Noetherian ring has finitely many associated primes, we obtain that any Noetherian reduced ring is normal if and only if it is integrally closed.
The last result of this section is an application of Proposition 2.3.3: we can decide, if the normalization of a ring is actually normal.

Proposition 2.3.4. *If R is a reduced ring with Noetherian normalization, then $\overline{R}$ is a normal ring.*
In particular: if R is reduced Noetherian and normalization-finite, then $\overline{R}$ is normal.

Proof. By Proposition 2.3.3, we need to show that $\overline{R}$ is integrally closed, has finitely many minimal prime ideals and is reduced. Since $\overline{R} \subseteq Q(R)$, we can apply Lemma 1.2.6 and obtain that $\overline{R}$ is reduced. By assumption, $\overline{R}$ is Noetherian, so $\mathrm{Min}(\overline{R})$ is finite by Lemma 1.1.5. Finally, lemma 2.1.7 states that $\overline{R}$ is integrally closed.
For the "in particular part", apply Lemma 2.2.7 to R. ■

Example 2.3.5. Let k be a field and $R = k[x,y]/\langle xy \rangle$. Then we have seen before that R is reduced and from Example 2.2.2, we can deduce that R is normalization-finite with: $\overline{R} = k[y] \times k[x]$. Hence, we can apply Proposition 2.3.4 and obtain that $\overline{R}$ is normal.

2.4 Normalization and completion - Excellent rings

The goal of this section is to give a short introduction to a class of rings whose completion commutes with normalization: so called *excellent rings*. Although the definition of these seems to be restricting, we will see that they are rather general and that most rings arising from algebraic geometry are indeed *excellent*. This is due to the facts, that fields are *excellent* and that *excellent rings* are closed under taking finitely generated algebras and localization.
The main result of the section will be of particular importance when we lift *invariants* to the completion of a ring in Chapter 4.

We start with the first condition on *excellent rings*. This concerns a particular subset of the spectrum of a ring: the *regular locus*. Recall, that the spectrum is a topological space, equipped with the Zariski Topology.

Definition 2.4.1. Let R be a Noetherian ring and $X = \mathrm{Spec}(R)$.

a) The **regular locus of** R is the set:

$$\mathrm{Reg}(X) = \{P \in X \mid R_P \text{ is regular}\}$$

b) R is **J-0**, if $\mathrm{Reg}(X)$ contains a non-empty open subset of X.

c) R is **J-1**, if $\mathrm{Reg}(X)$ is open in X.

d) R is **J-2**, if one of the equivalent conditions of Theorem 2.4.2 is satisfied.

Theorem 2.4.2. *Let R be a Noetherian ring. Then the following are equivalent:*

a) Any finitely generated R-algebra is J-1.

b) Any R-algebra which is finitely generated as R-module is J-1.

c) Let $P \in \mathrm{Spec}(R)$ and K a finite radical field extension of $\kappa(P)$. Then there exists an R-algebra S, which is finitely generated as R-module so that:

- $R/P \subseteq S \subseteq K$
- *S is J-0*
- $Q(S) = K$

Proof. See [Mat80, Theorem 73]. ■

Condition c) of Theorem 2.4.2 can be localized since for a multiplicatively closed subset W of R and a prime ideal P of R with $W \cap P = \emptyset$, we have that $\kappa(W^{-1}P) = \kappa(P)$. Hence, we obtain as a consequence, that J-2 rings are closed under localization. We also get that finitely generated algebras over J-2 rings are again J-2 by condition a) of Theorem 2.4.2.

Remark 2.4.3. Let R be a J-2 ring, W a multiplicatively closed subset of R and S a finitely generated R-algebra. Then $W^{-1}R$ is J-2 and S is J-2.

In order to formulate the second condition on *excellent rings*, we have to take a look at regular algebras over fields, which stay regular, when we extend the ground field:

Definition 2.4.4. Let R and S be Noetherian rings.

a) If R contains a field k, then R is called **geometrically regular** if $R \otimes_k L$ is regular for any finitely generated field extension L/k.

b) A ring homomorphism $R \to S$ is said to be **regular** if it is flat and for any $P \in \mathrm{Spec}(R)$ the fibre $S \otimes_R \kappa(P)$ is geometrically regular over $\kappa(P)$.

c) R is called a **G-ring** if for any prime ideal P of R, the map

$$R_P \to \widehat{R_P}$$

is regular. Here, the completion is built with respect to the P_P-adic topology on R_P.

Like for the J-2 condition we want, that G-rings are closed under localization and passing to finitely generated algebras. Therefore we will need the following theorem:

Theorem 2.4.5. *Let R be a G-ring and S an R-algebra essentially of finite type. Then S is a G-ring.*

Proof. This is [Stacks, Tag 07PV] ■

The last condition demands, that maximal prime chains in a ring always have same length:

Definition 2.4.6. Let R be any ring.

a) Then R is called **catenary**, if for all pair of prime ideals $Q \subseteq P$, the height of P/Q is finite and equal to the length of any maximal prime chain between Q and P.

b) If R is Noetherian and any finitely generated R-algebra is catenary, then R is called **universally catenary**.

It is immediately clear from the definition, that universally catenary rings are closed under passing to a finitely generated algebra. This also works for localization:

Lemma 2.4.7. *Any localization of a universally catenary ring is universally catenary.*

Proof. [Stacks, Tag 00NJ] ■

Corollary 2.4.8. *If R is an universally catenary ring and S an R-algebra, essentially of finite type, then S is universally catenary.*

Proof. Combine Lemma 2.4.7 and Definition 2.4.6. ■

Proposition 2.4.9. *Any field is universally catenary.*

Proof. See [Eis95, Corollary 13.6]. ■

Now we can state the definition of *excellent rings*:

Definition 2.4.10. A ring R is called **excellent ring** if it satisfies the following conditions:

- R is Noetherian
- R is J-2
- R is a G-ring
- R is universally catenary

The first examples of excellent rings will be fields. Since we have stated the closedness-properties for the conditions in the above definition, we will immediately get, that any algebra, essentially of finite type over a field, is excellent.

Lemma 2.4.11. *Any field is excellent.*

Proof. Let k be a field, then it is clear that k is Noetherian and universally catenary by 2.4.9. Now we show that k is J-2 by using condition c) of Theorem 2.4.2:
Let L be a finite radical extension of $k = \kappa(\langle 0 \rangle)$. Then, L is a finite dimensional k-vector space and we have $k \subseteq L$. The spectrum of L is just $\langle 0 \rangle$. Hence, $\mathrm{Reg}(\mathrm{Spec}(L)) = \mathrm{Spec}(L)$ since L is regular and therefore L is J-0. Altogether, k is J-2.
The field k is also a G-ring: the only map, we have to consider is the identity $k \to k$ and this is regular. ■

Proposition 2.4.12. *Let R be an excellent ring and S an R-algebra essentially of finite type. Then S is excellent.*

Proof. The J-2 property is preserved under passing to S by Remark 2.4.3, S is a G-ring by Theorem 2.4.5 and S is universally catenary by Corollary 2.4.8. Hence, S is excellent. ■

Remark 2.4.13. Matsumura also states in Chapter 13 of his book [Mat80], that formal power series rings over fields, as well as convergent power series rings over $\mathbb{R}$ or $\mathbb{C}$ are excellent. With this, Lemma 2.4.11 and Proposition 2.4.12, we obtain a whole variety of excellent rings:

- An algebra, essentially of finite type over a field, is excellent.
- If k is a field, then $k[[\underline{x}]]/I$ is excellent.
- Quotients of the convergent power series ring like $\mathbb{C}\{\underline{x}\}/I$ are excellent.

The final result of this section is, that excellent rings behave well with respect to completion and normalization:

Theorem 2.4.14. *If R is a reduced excellent ring, then:*

a) The completion $\widehat{R}$ at any ideal of R is reduced.

b) R is normalization-finite.

c) If R is in addition semi-local, then we have for the $\mathfrak{m}$-adic completion, where $\mathfrak{m}$ denotes the Jacobson radical:

$$\overline{\widehat{R}} = \widehat{\overline{R}}$$

Proof. See [BDS14, Theorem 1.18]. ■

We finish this section with an example:

Example 2.4.15. Let k be a field, $R = k[x,y]/\langle xy\rangle$ and $S = R_{\langle \overline{x},\overline{y}\rangle}$. Then we know from previous examples, i.e. 2.1.16, that R is reduced and $\overline{R} = k[y] \times k[x]$. In Proposition 2.2.6, we have seen that $\overline{S}$ is the localization of $\overline{R}$ at $\langle \overline{x},\overline{y}\rangle$:

$$\overline{S} = \overline{R}_{\langle \overline{x},\overline{y}\rangle} = k[y]_{\langle y\rangle} \times k[x]_{\langle x\rangle}$$

Since S is essentially of finite type over k, it is excellent by Remark 2.4.13. So we can compute the completion of $\overline{S}$:
First, we note that $\widehat{S} = k[[x,y]]/\langle xy\rangle$ by Example 1.4.6 and Proposition 1.4.14. Then this is also reduced and the minimal primes are $\langle \overline{x}\rangle$ and $\langle \overline{y}\rangle$. Hence, we can compute the normalization by Proposition 2.1.15, and keeping Theorem 2.4.14 in mind, we obtain:

$$\widehat{\overline{S}} = \overline{\widehat{S}} = k[[y]] \times k[[x]]$$

2.5 Normalization and field extension

We will later compute *invariants* of algebras over the field $\mathbb{Q}$ and we would like to lift the results to algebras over $\mathbb{C}$. We therefore examine the interplay between field extension and normalization. Like in the preceding section, the final goal is a theorem, that allows us to swap field extension and normalization of an algebra over a perfect field.
Before we can state this, we need the notion of a *normal* ring homomorphism:

Definition 2.5.1. A ring homomorphism $R \to S$ is called **normal** if it is flat and if the ring $S \otimes_R L$ is normal for any $P \in \operatorname{Spec}(R)$ and any field extension L of $\kappa(P)$.

The idea of such ring homomorphisms is, that they preserve normality under base-change. The special case, where the base-change is actually a field extension, is treated in Theorem 2.5.3. But first, we have to state, when a field extension is normal:

Proposition 2.5.2. *If L/k is a separable field extension, then $k \to L$ is normal.*

Proof. The statement is part of [HS06, Proposition 19.1.1]. ■

Hence, we know how to generate normal field extensions - in particular, over a perfect field, any field extension is normal. So we may consider now the above mentioned idea of normal homomorphisms.

Theorem 2.5.3. *Let $k \to L$ be a normal field extension. If S is a normal k-algebra, the ring $S \otimes_k L$ is a normal L-algebra.*

Proof. This is a special case of [HS06, Theorem 19.4.2]. ■

This is a key fact in proving the final theorem of this section.

Theorem 2.5.4. *Let S be a reduced Noetherian normalization-finite algebra over a perfect field k. Then taking the integral closure commutes with field extensions:*

$$\overline{S} \otimes_k L = \overline{S \otimes_k L}$$

for any field extension L/k.
In particular: the statement holds for any reduced excellent algebra over a perfect field.

Proof. Let L be any field extension of the perfect field k. Then by definition, L is separable over k. Proposition 2.5.2 shows, that the field extension is even normal. The ring S is normalization-finite, so we may apply Proposition 2.3.4 and obtain that $\overline{S}$ is a normal k-algebra. Hence, we are in the following situation:

$$\begin{tikzcd}
\overline{S} \arrow[r] & \overline{S} \otimes_k L \\
k \arrow[u] \arrow[r, "\text{normal}"] & L \arrow[u]
\end{tikzcd}$$

We can apply Theorem 2.5.3 and obtain that $\overline{S} \otimes_k L$ is a normal L-algebra.

The extension $S \subseteq \overline{S}$ is clearly integral. Applying Lemma 2.1.6, with $T = S \otimes_k L$, we get that also $S \otimes_k L \subseteq \overline{S} \otimes_k L$ is an integral extension. Thus, we can extend the above picture:

$$\begin{tikzcd}
\overline{S} \arrow[r] & \overline{S} \otimes_k L \\
S \arrow[u, "\text{int}"] \arrow[r] & S \otimes_k L \arrow[u, "\text{int}"'] \\
k \arrow[u] \arrow[r, "\text{normal}"] & L \arrow[u]
\end{tikzcd}$$

We want to use Proposition 2.1.13 to prove that $\overline{S} \otimes_k L$ is the normalization of $S \otimes_k L$, so that we can deduce: $\overline{S} \otimes_k L = \overline{S \otimes_k L}$. Therefore we have to show that $S \otimes_k L$ is reduced, $\overline{S} \otimes_k L$ is integrally closed and that we have an inclusion $\overline{S} \otimes_k L \hookrightarrow Q(S \otimes_k L)$.

The normal algebra $\overline{S} \otimes_k L$ is integrally closed: normal rings are always integrally closed by Lemma 2.3.2.

We mentioned above that L/k is separable and since S is reduced, we may apply Lemma 1.3.4 and obtain that $S \otimes_k L$ is also reduced.

To show the inclusion, we first note that $\overline{S} \subseteq Q(S)$ and since L is flat over k by A.2.3, we also have: $\overline{S} \otimes_k L \subseteq Q(S) \otimes_k L$. So it is enough to embed $Q(S) \otimes_k L$ into $Q(S \otimes_k L)$.
Let S^{reg} and $(S \otimes_k L)^{reg}$ denote the sets of non-zero divisors of S and $S \otimes_k L$ and consider the canonical map $\varphi : S \to S \otimes_k L$. Then φ maps S^{reg} to $(S \otimes_k L)^{reg}$: Let b be a non-zero divisor of S, then $S \xrightarrow{\cdot b} S$ is injective and since L is flat over k, also the induced map $S \otimes_k L \xrightarrow{\cdot b \otimes id} S \otimes_k L$ is injective. But the latter map corresponds to multiplication by $b \otimes 1$.
Now set $W = \varphi(S^{reg}) = \left\{ s \otimes 1 \,\middle|\, s \in S^{reg} \right\}$. This is a multiplicatively closed subset of $S \otimes_k L$ and by the above considerations, we have: $W \subseteq (S \otimes_k L)^{reg}$. Hence, the natural map $W^{-1}(S \otimes_k L) \to Q(S \otimes_k L)$ is injective. Now it is left to show that $W^{-1}(S \otimes_k L) \cong Q(S) \otimes_k L$.
If we tensor the canonical embedding $S \to Q(S)$ by L over k, we get the embedding $S \otimes_k L \to Q(S) \otimes_k L$. This map sends W to the units of $Q(S) \otimes_k L$. By the universal property of localization, we obtain an injective map:

$$\begin{aligned} W^{-1}(S \otimes_k L) &\longrightarrow Q(S) \otimes_k L \\ \frac{a \otimes l}{b \otimes 1} &\longmapsto \frac{a}{b} \otimes l \end{aligned}$$

The map is also surjective: let $\sum \frac{a_i}{b_i} \otimes l_i \in Q(S) \otimes_k L$, then b_i is an element of S^{reg} and $b_i \otimes 1 \in W$. So we can construct the preimage: $\sum \frac{a_i \otimes l_i}{b_i \otimes 1}$. Hence, the map is an isomorphism and this completes the proof of the theorem. ∎

As an illustration, we consider our standard-example:

Example 2.5.5. Let $S = \mathbb{Q}[x,y]/\langle xy \rangle$. Then $S \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[x,y]/\langle xy \rangle$. Since we are in characteristic 0, the ground field is perfect and S is reduced excellent. Hence, we get by Theorem 2.5.4:

$$(\mathbb{Q}[y] \times \mathbb{Q}[x]) \otimes_{\mathbb{Q}} \mathbb{C} = \overline{S} \otimes_{\mathbb{Q}} \mathbb{C} = \overline{S \otimes_{\mathbb{Q}} \mathbb{C}} = \mathbb{C}[y] \times \mathbb{C}[x]$$

3. Derivations, Kähler differentials and universally finite differentials

A collection of the theory of Kähler differentials and related topics is given in this chapter. In Section 3.1, we begin with the notion of derivations. After this, we look at the module of Kähler differentials and state properties about its structure in certain situations. Section 3.2 treats differential algebras. These are graded algebras with a certain degree 1 map. Like for the Kähler differentials, we will also define an universal object here: the universal differential algebra. To obtain finitely generated objects, we will consider universally finite differentials in Section 3.3. We treat the algebra and the module case simultaneously: we will get an algebra and a module that are still universal but finitely generated. In the last section of this chapter, 3.4, we take a closer look at derivations and derivation modules: the main content is the behaviour of these modules under natural operations like localization or completion.

3.1 Kähler differentials

The very well-known concept of derivations in analysis can partly be translated to algebra: here we will take a look at maps that *behave like derivations*. Such maps will also be called *derivations* and we will see that they satisfy the product rule, the quotient rule and other rules that we already know. The module of *Kähler differentials* connects *derivations* with a universal property: we will be able to represent a given *derivation* as a linear map - so we actually linearize *derivations*, although they are non-linear over the ring, where they are defined. It will turn out that this universal property uniquely determines the module and that any ring has a module of *Kähler differentials*. In particular, it will often have a nice form but is not always finitely generated. Therefore, we will take a look at *universally finite differentials* in Section 3.3.

We begin with the definition of *derivations*, these are maps that satisfy the product rule.

Definition 3.1.1. Let R be a ring, S an R-algebra and M an S-module.

a) An R-linear map $d : S \to M$ is called R-**derivation** if it fulfils:

$$d(ab) = ad(b) + bd(a)$$

for all a, b in S. If there is no confusion, we just call d a **derivation** and write da for the image of a under d.

b) The set of all R-derivations from S to M is denoted by $\mathrm{Der}_R(S, M)$. This forms an S-module, as is easily checked. Therefore, it is called **module of derivations**. If $S = M$, then we just write: $\mathrm{Der}_R(S)$.

The fact that derivations satisfy the product rule already allows us to phrase some easy properties, known from analysis.

Lemma 3.1.2. *Let R be a ring, S an R-algebra, M an S-module and $d : S \to M$ a derivation. Then the following statements hold:*

a) $d(r) = 0$ for any $r \in R$ - Constants are mapped to zero.

b) $d(s^n) = ns^{n-1}d(s)$ - The power rule.

c) For $t \in S^*$ and $s \in S$, we have: $d(st^{-1}) = (t^{-1})^2(td(s) - sd(t))$ - The quotient rule.

Proof. This is an extract of [Kun86, 1.9]. ∎

With the help of part c) of Lemma 3.1.2, we can extend derivations to quotient fields:

Lemma 3.1.3. *If R is a ring and S an R-algebra, then any R-derivation $S \to S$ can be uniquely extended to an R-derivation $Q(S) \to Q(S)$.*
In particular: $\mathrm{Der}_R(S) \hookrightarrow \mathrm{Der}_R(Q(S))$.

Proof. Let $d : S \to S$ be a derivation. Setting

$$\begin{aligned} D : Q(S) &\longrightarrow Q(S) \\ \frac{a}{b} &\longmapsto \frac{bd(a) - ad(b)}{b^2}, \end{aligned}$$

we obtain an R-derivation and clearly $D|_S = d$. So we have to show that this is the only extension of d to $Q(S)$. Therefore, let D' be another extension. Then, for an element $\frac{a}{b}$ of $Q(S)$, we have by Lemma 3.1.2:

$$D'\left(\frac{a}{b}\right) = \frac{bD'(a) - aD'(b)}{b^2} = \frac{bd(a) - ad(b)}{b^2} = D\left(\frac{a}{b}\right)$$

∎

Not only rules, but also explicit derivations can be translated from analysis to algebra. For example, we just take the usual partial derivative and apply it to polynomials. This will give an important example of a derivation on the polynomial ring. It is even possible to extend this to power series:

Example 3.1.4. Let R be a ring and $S = R[x_1, \ldots, x_n]$ the polynomial ring over R. The **partial derivative** of a polynomial $f = \sum c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in S$ is defined by:

$$\frac{\partial f}{\partial x_i} = \sum \alpha_i c_\alpha x_1^{\alpha_1} \ldots x_i^{\alpha_i - 1} \ldots x_n^{\alpha_n}$$

This is again a polynomial and we obtain for any $i = 1, \ldots, n$ a derivation on the polynomial ring:

$$\begin{aligned} \frac{\partial}{\partial x_i} : S &\longrightarrow S, \\ f &\longmapsto \frac{\partial f}{\partial x_i} \end{aligned}$$

If $T = R[[x_1, \ldots, x_n]]$ is the ring of formal power series over R, then the partial derivatives induce derivations on T in the same way as on S.

We have seen in Lemma 3.1.2, that constants are mapped to zero under derivations. This also holds for idempotent elements:

Remark 3.1.5. [Eis95, Exercise 16.1] Let R be a ring and S an R-algebra, M an S-module and $d : S \to M$ a derivation. If $b \in S$ is idempotent, then $d(b) = 0$.

Proof. Using the product rule and the idempotency of b, we obtain: $d(b) = d(b^2) = 2bd(b)$.

$$\Rightarrow 0 = 2bd(b) - d(b) \text{ in M} \tag{3.1}$$

$$\overset{\cdot b}{\Rightarrow} 0 = 2b^2 d(b) - bd(b) = 2bd(b) - bd(b) = bd(b) \tag{3.2}$$

If we plug (3.2) into equation (3.1), we simply get: $0 = d(b)$. ■

Now that we have seen some properties and examples of derivations, we wish to gain deeper insight into the structure of the module of derivations. But we do not examine this module directly, we focus on the construction and properties of the so-called *differential module* and it turns out that the derivation module is just the dual of it.

Definition 3.1.6. Let R be a ring and S an R-algebra. The **module of (Kähler) differentials**, written $\Omega^1_{S/R}$, with **universal derivation** d is an S-module together with an R-derivation $d : S \to \Omega^1_{S/R}$, satisfying the following universal property:

For any S-module M and any R-derivation $\delta : S \to M$, there exists a unique S-linear map $\varphi : \Omega^1_{S/R} \to M$, so that the following diagram commutes:

$$\begin{tikzcd} & \Omega^1_{S/R} \arrow[dd, "\varphi"] \\ S \arrow[ur, "d"] \arrow[dr, "\delta"'] & \\ & M \end{tikzcd}$$

It is not yet clear, why such a module should exist for an arbitrary algebra S over a ring R. Nonetheless, we can state how it is related to the module of derivations and that it is unique:

Remark 3.1.7. Let R be a ring and S an algebra so that $\Omega^1_{S/R}$ exists. As an immediate consequence of the universal property, we obtain:

a) For each S-module M, we have a natural isomorphism:

$$\mathrm{Der}_R(S, M) \cong \mathrm{Hom}_S(\Omega^1_{S/R}, M)$$

and in particular: $\mathrm{Der}_R(S) \cong \mathrm{Hom}_S(\Omega^1_{S/R}, S)$.

b) The module $\Omega^1_{S/R}$ is unique.

c) If d is the universal derivation, then we get an equality:

$$\Omega^1_{S/R} = \langle ds \mid s \in S \rangle_S$$

since the module on the right-hand side already satisfies the universal property and is a submodule of $\Omega^1_{S/R}$.

Theorem 3.1.8. *For a ring R and an R-algebra S, the module of Kähler differentials exists.*

Proof. This is [Kun86, Theorem 1.19]. ■

Now we collect some statements about the structure of $\Omega^1_{S/R}$. Having geometry in mind, we should start with the case, where S is a polynomial ring over R. We will use the partial derivations, seen in Example 3.1.4, to construct a universal derivation and the module of Kähler differentials:

Proposition 3.1.9. *Let R be a ring and $S = R[x_1, \dots, x_n]$, then*

$$\Omega^1_{S/R} = \bigoplus_{i=1}^{n} S dx_i$$

and the universal derivation is given by: $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Proof. See [Eis95, Proposition 16.1]. ■

For describing the structure of the Kähler differentials in more general cases, it is practical to have the two fundamental sequences:

Proposition 3.1.10. *Let $R \to S \to T$ be ring homomorphisms. Then we get an exact sequence of T-modules:*

$$T \otimes_S \Omega^1_{S/R} \longrightarrow \Omega^1_{T/R} \longrightarrow \Omega^1_{T/S} \longrightarrow 0,$$

where the left-hand map takes $t \otimes ds$ to tds and the right-hand map sends dt to dt.

Proof. This is [Eis95, Proposition 16.2]. ■

Proposition 3.1.11. *Let R be a ring and $S \to T$ a surjective ring-homomorphism of R-algebras with kernel I, then we get an exact sequence of T-modules:*

$$I/I^2 \longrightarrow T \otimes_S \Omega^1_{S/R} \longrightarrow \Omega^1_{T/R} \longrightarrow 0,$$

where the left-hand map sends $\overline{f}$ to $1 \otimes df$ and the right-hand map sends $a \otimes db$ to adb.

Proof. See [Eis95, Proposition 16.3]. ■

As a consequence, we can describe the structure of quotients:

Corollary 3.1.12. *Let R be a ring and S an R-algebra. If I is an ideal of S and $T = S/I$, then:*

$$\Omega^1_{T/R} = \Omega^1_{S/R}/(dI + I\Omega^1_{S/R})$$

Proof. We get an epimorphism $S \to T$ and therefore by Proposition 3.1.11 the exactness of:

$$I/I^2 \xrightarrow{\varphi} T \otimes_S \Omega^1_{S/R} \longrightarrow \Omega^1_{T/R} \longrightarrow 0$$

Hence, $\Omega^1_{T/R} = \mathrm{Coker}(\varphi)$. But since

$$T \otimes_S \Omega^1_{S/R} = S/I \otimes_S \Omega^1_{S/R} = \Omega^1_{S/R}/I\Omega^1_{S/R}$$

and φ maps $\overline{f}$ to $1 \otimes df$, we get: $\mathrm{Coker}(\varphi) = \Omega^1_{S/R}/(dI + I\Omega^1_{S/R})$. ■

Kähler differentials can also deal with base changes from the underlying ring: If we have a ring R and an R-algebra S with differential module $\Omega^1_{S/R}$, then tensoring with another R-algebra T yields the module $\Omega^1_{S/R} \otimes_R T$. This is in fact the module of differentials of $S \otimes_R T$ over T.

Proposition 3.1.13. *Let R be a ring. Then for any R-algebras S and T, we have:*

$$\Omega^1_{S/R} \otimes_R T \cong \Omega^1_{(S \otimes_R T)/T}$$

Proof. See [Eis95, Proposition 16.4]. ■

In Proposition 3.1.9, we have considered the case where S is a polynomial ring over R. The next result shows, what happens if T is a polynomial ring over S, which is an arbitrary algebra over R:

Proposition 3.1.14. *Let R be a ring and S an arbitrary R-algebra. If $T = S[x_1, \ldots, x_n]$, then we have an isomorphism:*

$$\Omega^1_{T/R} \cong T \otimes_S \Omega^1_{S/R} \oplus \bigoplus_{i=1}^{n} T dx_i$$

Proof. This is [Eis95, Corollary 16.6]. ■

As an application of the preceding proposition, we may consider the following fact, that we will need in Chapter 4.

Lemma 3.1.15. *Let R be a ring, S and T R-algebras and $S \to T$ of finite type. If $\Omega^1_{S/R}$ is finitely generated over S, then $\Omega^1_{T/R}$ is finitely generated over T.*

Proof. By assumption, we can write: $T = S[\underline{x}]/I$, where $\underline{x} = x_1, \ldots, x_n$ and applying Proposition 3.1.14, we can deduce that

$$\Omega^1_{S[\underline{x}]/R} \cong S[\underline{x}] \otimes_S \Omega^1_{S/R} \oplus \bigoplus_{i=1}^{n} S[\underline{x}] dx_i$$

Since $\Omega^1_{S/R}$ is finitely generated over S, $\Omega^1_{S[\underline{x}]/R}$ is finitely generated over $S[\underline{x}]$. By Corollary 3.1.12, we know that $\Omega^1_{T/R}$ is a quotient of $\Omega^1_{S[\underline{x}]/R}$ and therefore finitely generated over T. ■

The next standard operation is localization. Kähler differentials also behave well with respect to this:

Proposition 3.1.16. *If R is a ring, S an R-algebra and W a multiplicatively closed subset of S, then:*

$$W^{-1}\Omega^1_{S/R} \cong \Omega^1_{W^{-1}S/R}$$

Proof. See [Eis95, Proposition 16.9]. ■

The last structural result for the differential module we consider, is the behaviour of $\Omega^1_{S/R}$ if S is actually a product of algebras. This is of particular interest in Chapter 4, since we want to describe the derivation module of the normalization of a reduced ring. The way to do this, is via considering the Kähler differentials of $\overline{S}$. But the normalization is a product of rings by Proposition 2.1.15.

Proposition 3.1.17. *[Eis95, Proposition 16.10] If R is a ring, $S_1, \ldots, S_r$ are R-algebras and $S = \prod S_i$, then: $\Omega_{S/R} = \prod \Omega_{S_i/R}$.*

Proof. We give a more detailed version of the proof of Eisenbud. The idea is to show that $\prod \Omega_{S_i/R}$ satisfies the universal property of the Kähler differential module. Let d_i be the universal derivation of $\Omega_{S_i/R}$ and set $d = (d_1, \ldots, d_r) : S \to \prod \Omega_{S_i/R}$. Obviously, this is a derivation, since any component is one.

If M is an arbitrary S-module and $\delta : S \to M$ is a derivation, then consider $M_i = e_i M$, where $e_i = (0, \ldots, 1, \ldots, 0) \in S$ is the "i-th unit vector". Note that the elements e_i are idempotent and that M_i is a module over S. Since S_i is a subring of S, M_i is also an S_i-module.

Now split the derivation δ using $\delta_i : S_i \to M_i$, $a \mapsto \delta(ae_i)$. This is in fact a map to M_i because $\delta_i(a) = \delta(ae_i) = \delta(ae_i^2) = e_i\delta(ae_i) + ae_i\delta(e_i) = e_i\delta(ae_i) \in M_i$, by the product rule and Remark 3.1.5. In particular, we obtain:

$$\delta_i(a) = \delta_i(a)e_i \tag{3.3}$$

The map δ_i is also a derivation:

$$\begin{aligned}\delta_i(ab) &= \delta(abe_i) = \delta(ae_ibe_i) = \delta(ae_i)be_i + \delta(be_i)ae_i \\ &= \delta_i(a)e_ib + \delta_i(b)e_ia \overset{(3.3)}{=} \delta_i(a)b + \delta_i(b)a\end{aligned}$$

By the universal property of $\Omega_{S_i/R}$, there exists a unique S_i-linear map $\varphi_i : \Omega_{S_i/R} \to M_i$ such that $\delta_i = \varphi_i \circ d_i$. By combining the φ_i, we get a product map $(\varphi_1, \ldots, \varphi_r) \in \prod \operatorname{Hom}_{S_i}(\Omega_{S_i/R}, M_i)$. Then by the 1:1-correspondence from Lemma A.3.4, there is a map $\varphi \in \operatorname{Hom}_S(\prod \Omega_{S_i/R}, M)$ with $\varphi(a_1, \ldots, a_r) = (\varphi_1(a_1), \ldots, \varphi_r(a_r))$. In fact, this is the map needed to satisfy the universal property:

$$\begin{aligned}\delta(a_1, \ldots, a_r) &= \delta\left(\sum_{i=1}^r a_ie_i\right) = \sum_{i=1}^r \delta(a_ie_i) \overset{(3.3)}{=} \sum_{i=1}^r \delta_i(a_i)e_i = \sum_{i=1}^r \varphi_i(d_i(a_i))e_i \\ &= (\varphi_1(d_1(a_1)), \ldots, \varphi_r(d_r(a_r))) = \varphi(d(a_1, \ldots, a_r))\end{aligned}$$

The map φ is also unique, since any $\varphi' \in \operatorname{Hom}_S(\prod \Omega_{S_i/R}, M)$ is again a product of maps $\varphi'_i \in \operatorname{Hom}_{S_i}(\Omega_{S_i/R}, M_i)$ by Lemma A.3.4. If φ' satisfies

$\delta = \varphi' \circ d$, then the φ'_i satisfy $\delta_i = \varphi'_i \circ d_i$. But the maps φ_i stem from the universal property - they are unique. This implies that $\varphi'_i = \varphi_i$ and thus $\varphi' = \varphi$. ■

There are many more structural results on Kähler differentials. But for our purpose, the facts collected in this sections, are enough. We finish this section with an example:

Example 3.1.18. Let k be a field and $R = k[x, y]$

a) Let $S = R/I$, where $I = \langle xy \rangle$. Then we get by Proposition 3.1.9, that $\Omega^1_{R/k} = Rdx \oplus Rdy$ and using Corollary 3.1.12, we can deduce:

$$\begin{aligned}\Omega^1_{S/k} &= \Omega^1_{R/k}/(dI + I\Omega^1_{R/k}) = (Rdx \oplus Rdy)/(dI + I(Rdx \oplus Rdy)) \\ &= (Sdx \oplus Sdy)/\,\overline{dI}\end{aligned}$$

And since dI is generated by $d(xy) = ydx + xdy$, we obtain:

$$\Omega^1_{S/k} = (Sdx \oplus Sdy)/\langle \overline{ydx + xdy} \rangle_S$$

b) We know from Example 2.1.16, that $\overline{S} = k[y] \times k[x]$. Hence, we may apply Proposition 3.1.17 to get:

$$\Omega^1_{\overline{S}/k} = \Omega^1_{k[y]/k} \times \Omega^1_{k[x]/k} = k[y]dy \times k[x]dx$$

c) More general, let $R = k[x_1, \ldots, x_n]$, I an ideal of R and $S = R/I$. Then we know, that I is finitely generated, $I = \langle f_1, \ldots, f_r \rangle$. Thus, dI is generated by $df_1, \ldots, df_r$. If we apply Proposition 3.1.9 and Corollary 3.1.12 as in a), we obtain:

$$\begin{aligned}\Omega^1_{S/k} &= \Omega^1_{R/k}/(dI + I\Omega^1_{R/k}) \\ &= \bigoplus_{i=1}^{n} Rdx_i \,/\langle df_1, \ldots, df_r, f_1, \ldots, f_r \rangle_R \\ &= \bigoplus_{i=1}^{n} Sdx_i \,/\langle \overline{df_1}, \ldots, \overline{df_r} \rangle_S\end{aligned}$$

d) If T is an algebra essentially of finite type over k, then we have that T is a localization of a ring $S = k[x_1, \ldots, x_n]/I$ at a multiplicatively closed subset W. Since we know the form of $\Omega^1_{S/k}$ by part c), we may apply Proposition 3.1.16 and obtain:

$$\Omega^1_{T/k} \cong W^{-1}\Omega^1_{S/k}$$

In particular: $\Omega^1_{T/k}$ is a finitely generated module over T.

e) If k has characteristic 0, then it is shown in [Kun86, Example 5.5], that $\Omega^1_{k[[x]]/k}$ is not finitely generated over $k[[x]]$. But this means, that our first guess, the module $k[[x]]dx$, cannot be the module of Kähler differentials. Nonetheless, we will see in Section 3.3, that this "natural" module still plays an important role.

3.2 Differential algebras

We have seen in the preceding section, that the Kähler differentials have a module structure and satisfy a certain universal property. Now we would like to have an object, that still satisfies a comparable universal property, but has an algebra structure. Therefore, we will first consider general *differential algebras*: graded algebras which carry a derivation as degree 1 map. Then we will formulate the universal property, leading us to the so called *universal differential algebras*. It turns out, that such a *differential algebra* has a particular nice form: it is the exterior algebra of a well-known module. For a SINGULAR-implementation of *universal differential algebras* over polynomial rings, see Appendix C.

Throughout this section, we may assume that R is a ring and S is an algebra over R.

Definition 3.2.1. A **differential algebra of** S/R is an associative graded, not necessarily commutative, S-algebra

$$\Omega = \bigoplus_{n\in\mathbb{N}} \Omega^n,$$

which carries an R-linear degree 1 map: $d : \Omega \to \Omega$, so that the following conditions are satisfied:

- $S = \Omega^0$ is in the center of Ω.
- Ω is generated by the elements ds, $s \in S$ as an S-algebra.
- For $s, t \in S$, we have: $d(st) = sdt + tds$.
- For $s, s_1, \ldots, s_m \in S$, we have: $d(sds_1 \ldots ds_m) = dsds_1 \ldots ds_m$.
- The product $dsds$ is zero for any $s \in S$.

The map d is called **differentiation** of Ω and the elements of Ω^n are called **n-forms**. Sometimes, we will denote the differential algebra by (Ω, d), to mention what the differentiation of Ω is.

With this, we can already state some useful rules:

Lemma 3.2.2. *Let (Ω, d) be a differential algebra of S/R with differentiation d. Then we can derive the following:*

a) $d|_S : S \to \Omega^1$ is an R-derivation.

b) Elements of the form $sds_1 \dots ds_m$ have degree m and an arbitrary $\omega \in \Omega$ is a finite sum of such elements.

c) If $\omega = \sum sds_1 \dots ds_m$, then $d\omega = \sum dsds_1 \dots ds_m$.

d) Ω is anticommutative: for any $\omega_m \in \Omega^m$ and $\omega_n \in \Omega^n$, we have:

$$\omega_m \cdot \omega_n = (-1)^{mn} \omega_n \cdot \omega_m$$

e) The differential algebra (Ω, d) induces a complex of R-modules:

$$\Omega : \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

Proof. This is part of [Kun86, 2.2]. ■

Before we move on to morphisms between differential algebras, we should have a look at some examples:

Example 3.2.3.

a) The trivial differential algebra is given by: $\Omega^0 = S$ and all other graded parts are defined to be zero: $\Omega^i = 0$ for $i > 0$. The differentiation of Ω is the zero map, then this is a differential algebra.

b) If $S = R[x_1, \dots, x_n]$, then $\Omega^1_{S/R} = \bigoplus_{i=1}^n Sdx_i$, as seen in Proposition 3.1.9. Now set $\Omega^j = \bigwedge^j \Omega^1_{S/R}$. This is a free S-module of rank $\binom{n}{j}$ and the basis is given by the elements:

$$dx_{\nu_1} \wedge \dots \wedge dx_{\nu_j}, \ 1 \le \nu_1 < \dots < \nu_j \le n$$

Define $\Omega = \bigoplus_{j \in \mathbb{N}} \Omega^j = \bigwedge \Omega^1_{S/R}$ and let d denote the universal derivation of S/R, then we want to extend d to Ω. Therefore, it is enough to lift it to homogeneous elements. Let $\omega_j \in \Omega^j$ be homogeneous, then we can write:

$$\omega_j = \sum_{1 \le \nu_1 < \dots < \nu_j \le n} s_{\nu_1 \dots \nu_j} dx_{\nu_1} \wedge \dots \wedge dx_{\nu_j}$$

Define $d : \Omega^j \to \Omega^{j+1}$ by

$$d\omega_j = \sum_{1 \le \nu_1 < \dots < \nu_j \le n} ds_{\nu_1 \dots \nu_j} \wedge dx_{\nu_1} \wedge \dots \wedge dx_{\nu_j} \in \Omega^{j+1}$$

Then this is indeed a differential algebra of S/R. This is proven in [Kun86, 2.5].

In fact, we will see, that such exterior algebras are the most universal differential algebras. Any other differential algebra of S/R will be a quotient of them.
Now let us define maps between differential algebras, that are compatible with the structure.

Definition 3.2.4. Let (Ω, d) be a differential algebra of S/R and (Ω', d') a differential algebra of T/R, where T is another R-algebra. A map $\varphi : \Omega \to \Omega'$ is called **homomorphism of differential algebras** if:

- φ is a homomorphism of R-algebras.
- $\varphi(\Omega^n) \subseteq \Omega'^n$ for any $n \in \mathbb{N}$.
- φ is compatible with differentiation: $\varphi \circ d = d' \circ \varphi$

We also write φ^n for $\varphi|_{\Omega^n} : \Omega^n \to \Omega'^n$. Note, that $\varphi^0 : S \to T$ is an homomorphism of R-algebras.

Remark 3.2.5. Let (Ω, d) be a differential algebra of S/R, (Ω', d') a differential algebra of T/R and $\varphi : \Omega \to \Omega'$ a homomorphisms of differential algebras.

a) The map φ is uniquely determined by its restriction to the degree 0 part: φ^0.
In particular: if $\rho : S \to T$ is a homomorphism of R-algebras, then there is at most one homomorphism of differential algebras $\varphi : \Omega \to \Omega'$, which satisfies: $\varphi^0 = \rho$.

b) The image of φ is a differential algebra of the R-algebra $\varphi^0(S)$.

c) $\mathrm{Ker}(\varphi) = I$ is a homogeneous, two-sided, differentially closed ($dI \subseteq I$) ideal of Ω.

Proof. Let $\omega \in \Omega$ be an arbitrary element. Using Lemma 3.2.2, we can write:

$$\omega = \sum s ds_1 \dots ds_m$$

Now we apply φ and make use of the fact, that we deal with a homomorphism of differential algebras:

$$\begin{aligned}\varphi(\omega) &= \sum \varphi(s)\varphi(ds_1) \dots \varphi(ds_m) \\ &= \sum \varphi(s) d'\varphi(s_1) \dots d'\varphi(s_m) \\ &= \sum \varphi^0(s) d'\varphi^0(s_1) \dots d'\varphi^0(s_m)\end{aligned}$$

Hence, φ is only determined by its degree 0 part. This proves part a). Statement b) and c) are part of [Kun86, 2.8]. ∎

Now that we know that the Kernel of a homomorphism of differential algebras $\varphi : \Omega \to \Omega'$ is a homogeneous, differentially closed ideal, denoted by I, we may consider the object Ω/I. This is an algebra over $S/(I \cap S)$ and it is naturally graded: its homogeneous parts are $\Omega^n/(I \cap \Omega^n)$. The differentiation d of Ω induces a map $\overline{d} : \Omega/I \to \Omega/I$, $\overline{\omega} \mapsto \overline{d\omega}$. This is well-defined since I is differentially closed and, in fact, the pair $(\Omega/I, \overline{d})$ also satisfies the other conditions of Definition 3.2.1:

Proposition 3.2.6. *Let I be a homogeneous right (left) ideal of the differential algebra (Ω, d) over S. Then:*

a) I is a two-sided ideal in Ω.

b) If I is, in addition, differentially closed, then we have that Ω/I with the induced map $\overline{d} : \Omega/I \to \Omega/I$ is a differential algebra of $S/(I \cap S)$ over R.

c) The canonical map $\epsilon : \Omega \to \Omega/I$ is a homomorphism of differential algebras.

Proof. Since Ω is anticommutative by Lemma 3.2.2, Part a) is a special case of [Kun86, Lemma 2.3].
The statements b) and c) are part of [Kun86, Proposition 2.9]. ■

To define the universal property, which we mentioned at the beginning of this section, we will need a special class of homomorphisms. If $\rho : S \to T$ is a homomorphism of R-algebras, Ω a differential algebra of S/R and Ω' a differential algebra of T/R, then we have seen in Remark 3.2.5, that there exists at most one homomorphism of differential algebras $\varphi : \Omega \to \Omega'$, so that $\varphi^0 = \rho$. Such a map φ is called **ρ-homomorphism**. In the case, where $S = T$ and $\rho = id_S$, a ρ-homomorphism is also called **S-homomorphism**.

Lemma 3.2.7. *Let φ be a ρ-homomorphism between differential algebras (Ω, d) and (Ω', d'). Then:*

a) It satisfies the formula:

$$\varphi(\sum s ds_1 \dots ds_m) = \sum \rho(s) d'\rho(s_1) \dots d'\rho(s_m)$$

b) The map φ is surjective if and only if ρ is.
In particular: any S-homomorphism is surjective.

c) If Ω and Ω' are differential algebras of S/R and $\varphi : \Omega \to \Omega'$, $\psi : \Omega' \to \Omega$ are S-homomorphisms, then φ and ψ are isomorphisms, inverse to each other.

Proof. As in the proof of Remark 3.2.5, we can deduce:

$$\varphi(\sum sds_1 \ldots ds_m) = \sum \varphi^0(s)d'\varphi^0(s_1)\ldots d'\varphi^0(s_m)$$

And since φ is a ρ-homomorphisms, we get that $\varphi^0 = \rho$. This proves a).
To prove b), assume that ρ is a map from S to T. Any element of Ω' is a sum of elements of the form: $td't_1 \ldots d't_m$, where $t, t_i \in T$ by Lemma 3.2.2. Hence, the claim is a direct consequence of a).
For Part c), note that $\psi \circ \varphi : \Omega \to \Omega$ is an S-homomorphism. But id_Ω is also an S-homomorphism and therefore they have to be equal. Similar arguments work for $\varphi \circ \psi$. ∎

We know that, for an arbitrary differential algebra, there is at most one ρ-homomorphism. But we did not yet mention anything about the existence. The following universal property guarantees the existence of ρ-homomorphisms:

Definition 3.2.8. Let $(\Omega_{S/R}, d)$ be a differential algebra of S/R, that satisfies the following universal property:

For any homomorphism $\rho : S \to T$ of R-algebras and any differential algebra (Ω, δ) of T/R, there is a (unique) ρ-homomorphism $h : \Omega_{S/R} \to \Omega$. So we are in the following situation:

$$\begin{tikzcd}
\Omega_{S/R} \arrow[r, dotted, "\exists! h"] & \Omega \\
S \arrow[u] \arrow[r, "\rho"] & T \arrow[u]
\end{tikzcd}$$

Then $(\Omega_{S/R}, d)$ is called **universal differential algebra of S/R**.

Theorem 3.2.9. *For any algebra S/R, a universal differential algebra $\Omega_{S/R}$ exists.*

Proof. See [Kun86, Theorem 3.2]. ∎

Like for the Kähler differentials, we also get the uniqueness from the universal property:

Remark 3.2.10. The universal differential algebra is unique up to isomorphism.

Proof. If (Ω, δ) is another differential algebra of S/R, satisfying the universal property, then we get S-homomorphisms $h : \Omega_{S/R} \to \Omega$ and $g : \Omega \to \Omega_{S/R}$:

$$\begin{tikzcd}
\Omega_{S/R} \arrow[r, dotted, shift left, "\exists! h"] & \Omega \arrow[l, dotted, shift left, "\exists! g"] \\
S \arrow[u] \arrow[r, shift left, "id_S"] & S \arrow[u] \arrow[l, shift left, "id_S"]
\end{tikzcd}$$

But then we obtain by Lemma 3.2.7 that these two S-homomorphisms are inverse to each other. Hence, $\Omega_{S/R} \cong \Omega$. ■

We have seen in Proposition 3.2.6 that a quotient of a differential algebra by an ideal is again a differential algebra, as long as the ideal is homogeneous and differentially closed. A consequence of the universal property of $\Omega_{S/R}$ is that all differential algebras of S/R are quotients of $\Omega_{S/R}$ by such ideals.

Corollary 3.2.11. *Let Ω be any differential algebra of S/R, then we get an equality:*

$$\Omega_{S/R}/I = \Omega,$$

where I is a homogeneous differentially closed ideal of $\Omega_{S/R}$.

Proof. By the universal property, we get an S-homomorphism $h : \Omega_{S/R} \to \Omega$. This is a surjective map by Lemma 3.2.7 and the Kernel of h is a homogeneous differentially closed ideal of $\Omega_{S/R}$ by Remark 3.2.5. ■

The next step in this section is to show that universal differential algebras are just exterior algebras. But first, we consider a relation, that already holds for any differential algebra:

Proposition 3.2.12. *Let Ω be a differential algebra of S/R. Then there is a canonical epimorphism of graded S-algebras:*

$$\begin{aligned} \epsilon : \bigwedge \Omega^1 &\longrightarrow \Omega \\ a_1 \wedge \cdots \wedge a_m &\longmapsto a_1 \cdots a_m \end{aligned}$$

Proof. See [Kun86, Proposition 3.6]. ■

Now we want ϵ to be an isomorphism - this happens if the differential algebra is universal:

Proposition 3.2.13. *The universal differential algebra $\Omega_{S/R}$ of S/R is an exterior algebra:*

$$\Omega_{S/R} = \bigwedge \Omega^1_{S/R}$$

Proof. This is part of [Kun86, Proposition 3.8]. ■

By Lemma 3.2.2, we know that the differentiation d of $\Omega_{S/R}$, restricted to the degree 0 part, is a derivation: $d|_S : S \to \Omega^1_{S/R}$. Using the universal property of $\Omega_{S/R}$ one can show, that $d|_S$ is universal and that $\Omega^1_{S/R}$ is the module of Kähler differentials. Hence, the degree 1 part of $\Omega_{S/R}$ and the Kähler differentials do not only share notation, they really coincide.

Proposition 3.2.14. *Let $(\Omega_{S/R}, d)$ be the universal differential algebra of S/R. Then the R-derivation $d|_S : S \to \Omega^1_{S/R}$ is universal and $\Omega^1_{S/R}$ is the module of Kähler differentials.*
In particular, the universal differential algebra is the exterior algebra of the module of Kähler differentials.

Proof. This is [Kun86, Proposition 3.9]. ∎

With this proposition, we can easily construct the universal differential algebra, as far as we know the module of Kähler differentials.

Example 3.2.15.

a) In Example 3.2.3, we have already constructed $\bigwedge \Omega^1_{S/R}$, where $S = R[x_1, \ldots, x_n]$. Now we know, that this is the universal differential algebra $\Omega_{S/R}$. In this case, it is free of rank $\sum_{j=0}^{n} \binom{n}{j} = 2^n$.

b) In Example 3.1.18, we have constructed the module of Kähler differentials $\Omega^1_{S/k}$ in the case $R = k[x_1, \ldots, x_n]$, $S = R/I$ and k a field. If $I = \langle f_1, \ldots, f_r \rangle$, then:

$$\begin{aligned} \Omega_{S/k} = \bigwedge \Omega^1_{S/k} &= \bigwedge \left(\bigoplus_{i=1}^{n} R dx_i \,/ \langle df_1, \ldots, df_r, f_1, \ldots, f_r \rangle_R \right) \\ &= \bigwedge \left(\bigoplus_{i=1}^{n} S dx_i \,/ \langle \overline{df_1}, \ldots, \overline{df_r} \rangle_S \right) \end{aligned}$$

This will be a very helpful representation when implementing the universal differential algebra over a polynomial (quotient) ring.

3.3 Universally finite differentials

We have seen in Example 3.1.18 that not any module of Kähler differentials is finitely generated. But if we want to do computations, it would surely be helpful to have finitely generated modules that still satisfy a universal property like the Kähler differentials do. We will define this universal property and state the existence of such modules - they are called *universally finite modules of differentials.* But first, we do this at the level of differential algebras: we will define a universal property that contains a certain finiteness condition. This yields the notion of *universally finite differential algebras.* The degree 1 part of such an algebra will be the *universally finite module of differentials.*
In addition, we will state the fundamental sequences for *universally finite modules of differentials* and examine the behaviour of this module with respect to completion.

We want to deal with finitely generated differential algebras. Therefore, we should examine when such an algebra is actually finitely generated:

Lemma 3.3.1. *Let S/R be an algebra and (Ω, d) a differential algebra of S/R. Then the following equivalence holds:*

$$\Omega \textit{ is a finitely generated } S\textit{-module} \Leftrightarrow \Omega^1 \textit{ is a finitely generated } S\textit{-module}$$

Proof. Assume Ω is finitely generated. Then $\Omega = \langle \omega_1, \ldots, \omega_k \rangle_S$. Pick the generators ω_i that are of degree 1. These generate Ω^1 since Ω is graded.
For the converse direction, let $\Omega^1 = \langle t_1, \ldots, t_l \rangle_S$. By Lemma 3.2.2, we know that t_i is a sum of elements $s'_j ds_j$, where $s'_j, s_j \in S$. Hence, we may assume, that $t_i = ds_i$ for an $s_i \in S$. The same lemma implies that $\Omega^m = \langle ds_{\nu_1} \cdots ds_{\nu_m} | 1 \leq \nu_j \leq l \rangle_S$. Hence, any graded part is finitely generated. It is left to show that there are only finitely many non-zero graded parts: consider Ω^m, where $m > l$. Then the generators are products $ds_{\nu_1} \cdots ds_{\nu_m}$, where $1 \leq \nu_j \leq l$. In this product, a factor occurs twice. Using the anticommutativity of Ω, which we have by Lemma 3.2.2 and the fact that $dsds = 0$ for any $s \in S$, we have $ds_{\nu_1} \cdots ds_{\nu_m} = 0$ and thus $\Omega^m = 0$. ■

Now we formulate a universal property, similar to that of the universal differential algebra: the difference will be the integrated finiteness.

Definition 3.3.2. Let S/R be an algebra and $(\widetilde{\Omega}_{S/R}, d)$ a differential algebra of S/R. Then $\widetilde{\Omega}_{S/R}$ is called **universally finite** if it is finitely generated as S-module and for each finitely generated differential algebra (Ω, δ) of S/R, there is a (unique) S-homomorphism $\widetilde{\Omega}_{S/R} \to \Omega$.
Like for the universal differential algebra, we get uniqueness up to isomorphism and we denote the universally finite differential algebra by $\widetilde{\Omega}_{S/R}$.

Since S-homomorphisms are always surjective, we could rephrase the universal property: any finitely generated differential algebra of S/R is a quotient of $\widetilde{\Omega}_{S/R}$ by a homogeneous differentially closed ideal. This is similar to Corollary 3.2.11.

Obviously, the universal differential algebra $\Omega_{S/R}$ satisfies the universal property of $\widetilde{\Omega}_{S/R}$ if it is finitely generated. Hence, we obtain:

Remark 3.3.3. For an algebra S/R, the following holds: if $\Omega_{S/R}$ is finitely generated, then: $\widetilde{\Omega}_{S/R} = \Omega_{S/R}$.

We have already seen examples for universally finite differential algebras: the algebras constructed in 3.2.15 are finitely generated. Hence, they are also universally finite. But we shall consider an example, where the universally finite differential algebra does not exist.

Example 3.3.4. If S/R is an algebra and S is a field, then $\widetilde{\Omega}_{S/R}$ exists if and only if $\Omega_{S/R}$ is finitely generated.
For the proof, we refer to [Kun86, Example 11.2].

Now that we have done the step from differential algebras to finitely generated differential algebras, we should also do this for modules of differentials: we integrate a finiteness condition into the universal property of Kähler differentials:

Definition 3.3.5. Let R be a ring and S an R-algebra. Suppose, we have a finitely generated S-module $\widetilde{\Omega}^1_{S/R}$ and an R-derivation $d : S \to \widetilde{\Omega}^1_{S/R}$, so that the pair $(d, \widetilde{\Omega}^1_{S/R})$ satisfies the following universal property:

For any finitely generated S-module M and any R-derivation $\delta : S \to M$, there exists a unique S-linear map $\varphi : \widetilde{\Omega}^1_{S/R} \to M$ so that the following diagram commutes:

$$\begin{tikzcd} & \widetilde{\Omega}^1_{S/R} \arrow[d, "\varphi"] \\ S \arrow[ur, "d"] \arrow[r, "\delta"'] & M \end{tikzcd}$$

Then $\widetilde{\Omega}^1_{S/R}$ is called **universally finite module of differentials** and d is called **universally finite derivation**.

Remark 3.3.6. Let R be a ring and S an R-algebra.

a) If the module of Kähler differentials is finitely generated, then it is clear that we have $\Omega^1_{S/R} = \widetilde{\Omega}^1_{S/R}$.

b) The universal property of the universally finite module of differentials can also be translated to another useful fact - for any finitely generated S-module M, we get an isomorphism of S-modules:

$$\mathrm{Der}_R(S, M) \cong \mathrm{Hom}_S(\widetilde{\Omega}^1_{S/R}, M)$$

In particular: $\mathrm{Der}_R(S) \cong \mathrm{Hom}_S(\widetilde{\Omega}^1_{S/R}, S)$.

Any module of Kähler differentials which is finitely generated is an example for a universally finite module of differentials. We have seen modules like this in Example 3.1.18. But we have also seen that for the power series ring over a field of characteristic 0, the module of Kähler differentials is not finitely generated. However, the universally finite module of differentials exists and we will see later that its form comes from a natural operation:

Lemma 3.3.7. *Let k be a field and $R = k[[x_1, \dots, x_n]]$. Then $\widetilde{\Omega}^1_{R/k}$ exists and:*

$$\widetilde{\Omega}^1_{R/k} = \bigoplus_{i=1}^{n} R dx_i,$$

where d is given by $f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i$.

The proof will be given later, when we consider the completion of the universally finite module of differentials.
A similar statement also holds for the convergent power series ring:

Lemma 3.3.8. *Let k be a field with a valuation and $R = k\{x_1, \dots, x_n\}$. Then $\widetilde{\Omega}^1_{R/k}$ exists and:*

$$\widetilde{\Omega}^1_{R/k} = \bigoplus_{i=1}^{n} R dx_i,$$

where d is given by $f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i$.

Proof. See [Sch70, Satz 5.1]. ∎

In Proposition 3.2.14, it was shown that the first graded part of $\Omega_{S/R}$ is the module of Kähler differentials. Therefore, it is not surprising that an analogue statement also holds for $\widetilde{\Omega}_{S/R}$ and the universally finite module of differentials:

Proposition 3.3.9. *Let S/R be an algebra. Then $\widetilde{\Omega}_{S/R}$ exists if and only if $\widetilde{\Omega}^1_{S/R}$ exists. In this case, the degree 1 part of $\widetilde{\Omega}_{S/R}$ is $\widetilde{\Omega}^1_{S/R}$.*

Proof. This is a consequence of [Kun86, Proposition 11.5]. ∎

Now that we have shown the relation between the universally finite differential algebra and the universally finite module of differentials, we focus on the latter. For the module of Kähler differentials, there are two fundamental sequences we have already seen in Section 3.1. These sequences translate to the current case:

Proposition 3.3.10. *Let $R \to S \to T$ be ring homomorphisms. If $\widetilde{\Omega}^1_{T/R}$ exists, then $\widetilde{\Omega}^1_{T/S}$ exists and we get a canonical exact sequence of T-modules:*

$$T \otimes_S \Omega^1_{S/R} \longrightarrow \widetilde{\Omega}^1_{T/R} \longrightarrow \widetilde{\Omega}^1_{T/S} \longrightarrow 0$$

Proof. See [Kun86, Proposition 11.17]. ∎

Proposition 3.3.11. *Let R be a ring and $S \to T$ a surjective ring-homomorphism of R-algebras with kernel I. If $\widetilde{\Omega}^1_{S/R}$ exists, then $\widetilde{\Omega}^1_{T/R}$ exists and we get an exact sequence of T-modules:*

$$I/I^2 \longrightarrow T \otimes_S \widetilde{\Omega}^1_{S/R} \longrightarrow \widetilde{\Omega}^1_{T/R} \longrightarrow 0$$

Proof. See [Kun86, Corollary 11.10]. ■

Corollary 3.3.12. *Let R be a ring, S an R-algebra, I an ideal of S and $T = S/I$. If $\widetilde{\Omega}^1_{S/R}$ exists, then $\widetilde{\Omega}^1_{T/R}$ exists and:*

$$\widetilde{\Omega}^1_{T/R} = \widetilde{\Omega}^1_{S/R}/(dI + I\widetilde{\Omega}^1_{S/R})$$

Proof. By Proposition 3.3.11, the universally finite module of differentials $\widetilde{\Omega}^1_{T/R}$ exists and we have the exact sequence. Now the proof is similar to Corollary 3.1.12. ■

The following important statement about the universally finite module of differentials and its corollary will be very useful when dealing with derivations of the normalization of an algebra.

Theorem 3.3.13. *Let R be a ring and S a Noetherian R-algebra so that $\widetilde{\Omega}^1_{S/R}$ exists. Let T be a finitely generated S-algebra, then $\widetilde{\Omega}^1_{T/R}$ also exists.*

Proof. This is a consequence of [Kun86, Proposition 11.9]. ■

Corollary 3.3.14. *Let R be a ring and S a reduced Noetherian normalization-finite R-algebra. If $\widetilde{\Omega}^1_{S/R}$ exists, then $\widetilde{\Omega}^1_{\overline{S}/R}$ exists.*

Proof. The normalization $\overline{S}$ is a finitely generated S-module. Hence, also a finitely generated S-algebra. Therefore, we may apply Theorem 3.3.13. ■

The last consideration in this section is the completion of a differential algebra. This will again be a differential algebra and under suitable assumptions, completion and differentials "commute".

Remark 3.3.15. Assume that R is a ring and S a Noetherian R-algebra. Let (Ω, d) be a finitely generated differential algebra of S/R and $\mathfrak{q}$ an ideal of S. For the $\mathfrak{q}$-adic completion $\widehat{\Omega}$, we have the following:

a) $\widehat{\Omega} = \bigoplus_{n\in\mathbb{N}} \widehat{\Omega^n}$, so $\widehat{\Omega}$ is a graded $\widehat{R}$-algebra.

b) The map $\widehat{d} : \widehat{\Omega} \to \widehat{\Omega}$ is an R-linear map of degree 1.

c) The module $(\widehat{\Omega}, \widehat{d})$ is a differential algebra of $\widehat{S}/R$.

Proof. This is explained in [Kun86, Remark 12.1]. ■

If k is a field of characteristic 0, then we know that $\Omega^1_{k[x]/k} = k[x]dx$. If we complete this at the maximal ideal $\langle x\rangle$, we obtain:

$$\widehat{\Omega^1_{k[x]/k}} = k[[x]]\widehat{d}x$$

But this finitely generated module cannot be the module of Kähler differentials of $k[[x]]/k$. Hence, the module of differentials is not preserved under completion. Nevertheless, we get a comparable result:

Proposition 3.3.16. *Let R be a ring, S a Noetherian R-algebra, $\mathfrak{q}$ an ideal of S and $\widehat{S}$ the $\mathfrak{q}$-adic completion of S.*

a) If $\Omega_{S/R}$ is finitely generated over S, then $\widetilde{\Omega}_{\widehat{S}/R}$ exists and we have:

$$\widetilde{\Omega}_{\widehat{S}/R} = \widehat{\Omega_{S/R}}$$

b) If $\Omega^1_{S/R}$ is finitely generated over S, then $\widetilde{\Omega}^1_{\widehat{S}/R}$ exists and we have:

$$\widetilde{\Omega}^1_{\widehat{S}/R} = \widehat{\Omega^1_{S/R}}$$

Proof. For Part a), we refer to [Kun86, Corollary 12.5].
To prove part b), we may use Lemma 3.3.1 and obtain that $\Omega_{S/R}$ is finitely generated. Hence, we apply a) and get the equality of $\widetilde{\Omega}_{\widehat{S}/R}$ and $\widehat{\Omega_{S/R}}$. Splitting the differential algebras into their graded parts using Remark 3.3.15, we can deduce:

$$\bigoplus \widetilde{\Omega}^j_{\widehat{S}/R} = \widetilde{\Omega}_{\widehat{S}/R} = \widehat{\Omega_{S/R}} = \bigoplus \widehat{\Omega^i_{S/R}}$$

Then the graded parts of degree 1 have to be equal. ∎

With this proposition we can prove Lemma 3.3.7:

Proof. Consider $R' = k[x_1, \ldots, x_n]$ and let $\mathfrak{m}$ be the maximal ideal generated by $x_1, \ldots, x_n$. By Example 1.4.6, we know that the $\mathfrak{m}$-adic completion of R' is the power series ring R. We also know from Proposition 3.1.9, that $\Omega^1_{R'/k} = \bigoplus_{i=1}^n R' dx_i$. Since this is finitely generated, we can apply Proposition 3.3.16 and Proposition 1.4.12:

$$\widetilde{\Omega}^1_{R/k} = \widehat{\Omega^1_{R'/k}} = \left(\bigoplus_{i=1}^n R' dx_i\right)^{\widehat{}} = \bigoplus_{i=1}^n R dx_i$$

Note that we used the notation d for two derivations: d is the universal derivation of the polynomial ring, sending a polynomial to the sum of its partial derivatives and d is the universal derivation of the power series ring: it maps a power series to the sum of its partial derivatives. ∎

There is even a generalization of Proposition 3.3.16, where we do not need to assume the finiteness of the module of Kähler differentials but only the existence of the universally finite module of differentials. If we do so, we have to restrict to semi local algebras and a particular completion:

Proposition 3.3.17. *Let R be a ring, S a semi local Noetherian R-algebra with Jacobson radical $\mathfrak{m}$ and $\widehat{S}$ the $\mathfrak{m}$-adic completion of S.*

a) If $\widetilde{\Omega}_{S/R}$ exists, then $\widetilde{\Omega}_{\widehat{S}/R}$ exists and we have:

$$\widetilde{\Omega}_{\widehat{S}/R} = \widehat{\widetilde{\Omega}_{S/R}}$$

b) If $\widetilde{\Omega}^1_{S/R}$ exists, then $\widetilde{\Omega}^1_{\widehat{S}/R}$ exists and we have:

$$\widetilde{\Omega}^1_{\widehat{S}/R} = \widehat{\widetilde{\Omega}^1_{S/R}}$$

Proof. Part a) is [Kun86, Corollary 12.10].
For Part b), we first apply Proposition 3.3.9 and obtain the existence of $\widetilde{\Omega}_{S/R}$. Hence, we get by a) that $\widetilde{\Omega}_{\widehat{S}/R}$ exists and that it is the completion $\widehat{\widetilde{\Omega}_{S/R}}$. Both algebras are graded. Therefore, the degree 1 parts have to coincide. The first graded part of $\widetilde{\Omega}_{\widehat{S}/R}$ is $\widetilde{\Omega}^1_{\widehat{S}/R}$ by Proposition 3.3.9 and the degree 1 part of $\widehat{\widetilde{\Omega}_{S/R}}$ is $\widehat{\widetilde{\Omega}^1_{S/R}}$ by Remark 3.3.15. ■

3.4 Derivations

The starting point of Chapter 3 were derivations. We have seen their definition at the beginning of Section 3.1 and we know that we can identify derivation modules as homomorphism modules of Kähler differentials or of the universally finite module of differentials. In this section, we focus on derivation modules of type $\operatorname{Der}_R(S)$, where S/R is an algebra. With the aforementioned identification as homomorphism modules, we can think of $\operatorname{Der}_R(S)$ as dual module of $\Omega^1_{S/R}$ or $\widetilde{\Omega}^1_{S/R}$, provided it exists. Therefore, it is natural that properties of these two modules transfer to derivations: we will take a look at the behaviour under localization, completion and field extension.
At the end of this section, we consider *I-preserving* derivations. As the name indicates, these are derivations which map an ideal I to itself, so I is preserved. This notion will be helpful when we deal with *invariants* in Chapters 4 and 5.

We begin this section with an example that shows how the structure of the module of Kähler differentials influences the structure of the derivation module.

Example 3.4.1. Let k be a field and $R = k[x_1, \dots, x_n]$. We have seen before that $\Omega^1_{R/k} = \bigoplus_{i=1}^n R dx_i$. Hence, the dual module $\operatorname{Hom}_R(\Omega^1_{R/k}, R)$ is also free with a basis $\varphi_1, \dots, \varphi_n$ satisfying $\varphi_i(dx_j) = \delta_{ij}$. Since we have the isomorphism:

$$\begin{aligned} \operatorname{Hom}_R(\Omega^1_{R/k}, R) &\longrightarrow \operatorname{Der}_k(R) \\ \varphi &\longmapsto \varphi \circ d, \end{aligned}$$

the module $\mathrm{Der}_k(R)$ is also free with basis $\varphi_i \circ d$, where $i = 1, \dots, n$. Now let $f \in R$, then:

$$(\varphi_i \circ d)(f) = \varphi_i \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \varphi_i(dx_j) = \frac{\partial f}{\partial x_i}$$

Thus, the module of derivations $\mathrm{Der}_k(R)$ is the free module, generated by the partial derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.

Since we already know that the module of Kähler differentials commutes with localization, we would expect the same for its dual:

Proposition 3.4.2. *Let R be a ring, S a Noetherian R-algebra so that $\Omega^1_{S/R}$ is finitely generated and W a multiplicatively closed subset of S. Then:*

$$\mathrm{Der}_R(W^{-1}S) \cong W^{-1} \mathrm{Der}_R(S)$$

Proof. The module of Kähler differentials is compatible with localization by Proposition 3.1.16. By our assumptions, it is also finitely generated. Hence, it is finitely presented since S is Noetherian. If we now identify the derivation module as homomorphism module as in Remark 3.1.7, we may apply Proposition A.2.2, since localization is flat:

$$\begin{aligned} \mathrm{Der}_R(W^{-1}S) &\cong \mathrm{Hom}_{W^{-1}S}(\Omega^1_{W^{-1}S/R}, W^{-1}S) \\ &\cong \mathrm{Hom}_{W^{-1}S}(W^{-1}\Omega^1_{S/R}, W^{-1}S) \\ &\cong W^{-1} \mathrm{Hom}_S(\Omega^1_{S/R}, S) \\ &\cong W^{-1} \mathrm{Der}_R(S) \end{aligned}$$

■

With the arguments used in Example 3.4.1, one could also determine the structure of the derivation module over a localized polynomial ring. But there is an alternative way, that is, apply Proposition 3.4.2:

Example 3.4.3. Let W be a multiplicatively closed subset of $R = k[x_1, \dots, x_n]$. Then by Example 3.4.1:

$$\mathrm{Der}_k(R) = \bigoplus_{i=1}^{n} R \frac{\partial}{\partial x_i}$$

Since localization and direct sum commute, we obtain:

$$\mathrm{Der}_k(W^{-1}R) \cong W^{-1} \mathrm{Der}_k(R) = \bigoplus_{i=1}^{n} W^{-1}R \frac{\partial}{\partial x_i}$$

Under the assumption that $\Omega^1_{S/R}$ is finitely generated, we get a comparable behaviour, as in Proposition 3.4.2, with respect to completion of the derivation module. But we have seen, that the module of Kähler differentials is not always finitely generated. Hence, we state a second result, which covers the case when $\Omega^1_{S/R}$ may not be finitely generated but $\widetilde{\Omega}^1_{S/R}$ exists.

Proposition 3.4.4. *Let R be a ring, S a Noetherian R-algebra, $\mathfrak{q}$ an ideal of S and $\widehat{S}$ the $\mathfrak{q}$-adic completion of S. If $\Omega^1_{S/R}$ is a finitely generated S-module, then the derivation module is compatible with completion:*

$$\mathrm{Der}_R(\widehat{S}) \cong \widehat{\mathrm{Der}_R(S)}$$

Proof. By Proposition 3.3.16, we get that the $\widehat{S}$-module $\widetilde{\Omega}^1_{\widehat{S}/R}$ exists and is equal to $\widehat{\Omega^1_{S/R}}$. Hence, we can deduce, using Remark 3.3.6, Theorem 1.4.11 and Proposition A.2.2:

$$\begin{aligned}
\mathrm{Der}_R(\widehat{S}) &\cong \mathrm{Hom}_{\widehat{S}}(\widetilde{\Omega}^1_{\widehat{S}/R}, \widehat{S}) = \mathrm{Hom}_{\widehat{S}}(\widehat{\Omega^1_{S/R}}, \widehat{S}) \\
&\cong \mathrm{Hom}_{S\otimes_S \widehat{S}}(\Omega^1_{S/R} \otimes_S \widehat{S}, S \otimes_S \widehat{S}) \\
&\cong \mathrm{Hom}_S(\Omega^1_{S/R}, S) \otimes_S \widehat{S} \\
&\cong \mathrm{Der}_R(S) \otimes_S \widehat{S} \\
&\cong \widehat{\mathrm{Der}_R(S)}
\end{aligned}$$

■

Proposition 3.4.5. *Let R be a ring and S a semi local Noetherian R-algebra so that $\widetilde{\Omega}^1_{S/R}$ exists. Denote by $\mathfrak{m}$ the Jacobson radical of S, then we get for the $\mathfrak{m}$-adic completion:*

$$\mathrm{Der}_R(\widehat{S}) \cong \widehat{\mathrm{Der}_R(S)}$$

Proof. We may apply Proposition 3.3.17 and obtain that $\widetilde{\Omega}^1_{\widehat{S}/R}$ exists and $\widetilde{\Omega}^1_{\widehat{S}/R} = \widehat{\widetilde{\Omega}^1_{S/R}}$. Now the proof is similar to the proof of Proposition 3.4.4. ■

In the following proposition, we examine if the derivation module $\mathrm{Der}_k(S)$ for an k-algebra S is stable under field extension of k. This will work if we again assume that the module of Kähler differentials is finitely generated.

Proposition 3.4.6. *Let L/k be a field extension and S a Noetherian k-algebra so that $\Omega^1_{S/k}$ is finitely generated. Then we get for the derivation module of the L-algebra $S \otimes_k L$:*

$$\mathrm{Der}_L(S \otimes_k L) \cong \mathrm{Der}_k(S) \otimes_k L$$

Proof. The following is a consequence of Proposition 3.1.13 and Corollary A.2.4:

$$\begin{aligned}\operatorname{Der}_L(S \otimes_k L) &\cong \operatorname{Hom}_{S\otimes_k L}(\Omega^1_{(S\otimes_k L)/L}, S \otimes_k L)\\ &\cong \operatorname{Hom}_{S\otimes_k L}(\Omega^1_{S/k} \otimes_k L, S \otimes_k L)\\ &\cong \operatorname{Hom}_S(\Omega^1_{S/k}, S) \otimes_k L\\ &\cong \operatorname{Der}_k(S) \otimes_k L\end{aligned}$$

■

Like the module of Kähler differentials, the derivation module of a product algebra splits into factors.

Proposition 3.4.7. *Let R be a ring, $S_1, \ldots, S_r$ R-algebras and $S = \prod S_i$. Then we have an isomorphism $\operatorname{Der}_R(S) \cong \prod \operatorname{Der}_R(S_i)$.*

Proof. Let for simplicity $S = S_1 \times S_2$. Then by Proposition 3.1.17, we obtain: $\Omega_{S/R} = \Omega_{S_1/R} \times \Omega_{S_2/R}$. Now we may apply Lemma A.3.4:

$$\begin{aligned}\operatorname{Der}_R(S) &\cong \operatorname{Hom}_S(\Omega_{S/R}, S) = \operatorname{Hom}_{S_1\times S_2}(\Omega_{S_1/R} \times \Omega_{S_2/R}, S_1 \times S_2)\\ &\cong \operatorname{Hom}_{S_1}(\Omega_{S_1/R}, S_1) \times \operatorname{Hom}_{S_2}(\Omega_{S_2/R}, S_2) \cong \operatorname{Der}_R(S_1) \times \operatorname{Der}_R(S_2)\end{aligned}$$

The result follows by induction. ■

For the end of this section, let k denote a field. The derivations that we will consider now preserve a certain ideal I of an k-algebra R. These special derivations form a submodule of $\operatorname{Der}_k(R)$ and, we will see that they represent the derivations of R/I in a nice way, if R is a polynomial ring or a localization of a polynomial ring. Later, we will need this notion to actually compute *invariants*.

Definition 3.4.8. Let R be an k-algebra and I an ideal of R. A derivation $\delta \in \operatorname{Der}_k(R)$ is called ***I*-preserving** if $\delta(I) \subseteq I$. The set of all I-preserving derivations is denoted by:

$$\mathrm{D}_I(R) = \{\delta \in \operatorname{Der}_k(R) \mid \delta(I) \subseteq I\}$$

Note: $\mathrm{D}_I(R)$ is a submodule of $\operatorname{Der}_k(R)$ and we always have: $I \operatorname{Der}_k(R) \subseteq \mathrm{D}_I(R)$.

The inclusion mentioned in the definition can be both: strict or an equality of submodules:

Example 3.4.9. Let $R = k[x, y]$.

a) Set $I = \langle x \rangle$, then we may look for a nice representation of all I-preserving derivations:

- The partial derivation $\frac{\partial}{\partial y}$ is I-preserving: if $gx \in I$, then $\frac{\partial}{\partial y}(gx) = x\frac{\partial}{\partial y}(g) \in I$.
- The derivation $x\frac{\partial}{\partial x}$ is also I-preserving.

Now let $\delta \in \mathrm{D}_I(R)$, then $\delta(x) \in I$. Since we know that $\mathrm{Der}_k(R) = R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y}$ by Example 3.4.1, we may write $\delta = g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y}$. Hence, we obtain: $\delta(x) = g_1$ is in I. Therefore, $\delta \in \langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$ and altogether:

$$\mathrm{D}_I(R) = \langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle_R \supsetneq I\,\mathrm{Der}_k(R),$$

where the strict inequality holds, since $\frac{\partial}{\partial y}$ is not in $I\,\mathrm{Der}_k(R)$.

b) Set $\mathfrak{m} = \langle x, y\rangle$. Then we actually get an equality:

$$\mathrm{D}_{\mathfrak{m}}(R) = \mathfrak{m}\,\mathrm{Der}_k(R)$$

Therefore, consider $\delta \in \mathrm{D}_{\mathfrak{m}}(R) \subseteq \mathrm{Der}_k(R)$. Then we may write $\delta = g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y}$. Since δ is $\mathfrak{m}$-preserving, we obtain:

$$\begin{aligned} g_1 &= \delta(x) \in \mathfrak{m} \\ g_2 &= \delta(y) \in \mathfrak{m} \end{aligned}$$

Hence, $\delta \in \mathfrak{m}\,\mathrm{Der}_k(R)$.

The next proposition shows that minimal primes in reduced rings are preserved by any derivation. This is in particular interesting, if we have finitely many minimal primes, as we will see in Section 4.4.

Proposition 3.4.10. *Let S be a reduced algebra over a ring R and δ any derivation in $\mathrm{Der}_R(S)$. Then δ leaves any minimal prime ideal P of S invariant:*

$$\delta \in \mathrm{D}_P(S) \text{ for any } P \in \mathrm{Min}(S)$$

Proof. See [CL91, p. 614]. ∎

Now that we have seen some examples, we focus on the representation-property of I-preserving derivations. This will be an identification, which allows us to represent the derivations of R/I as a quotient of $\mathrm{D}_I(R)$.

Lemma 3.4.11. *Let k be a field and R an k-algebra so that $\Omega^1_{R/k}$ is free and I an ideal of R. Then we have an isomorphism:*

$$\mathrm{Der}_k(R/I) \cong \mathrm{D}_I(R)/I\,\mathrm{Der}_k(R)$$

Proof. The proof is a generalization of [BS95, Lemma 2.1.2].
We can write $\Omega^1_{R/k} = \bigoplus_i R dx_i$. Hence, we get for the dual: $\mathrm{Der}_k(R) = \bigoplus_i R\frac{\partial}{\partial x_i}$, where $\frac{\partial}{\partial x_i}$ denotes the dual basis, satisfying the property:

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Define the map

$$\begin{aligned} \varphi : \mathrm{D}_I(R) &\longrightarrow \mathrm{Der}_k(R/I) \\ \delta &\longmapsto \overline{\delta}, \end{aligned}$$

where $\overline{\delta}(\overline{x}) = \overline{\delta(x)}$. Then φ is well-defined since for any I-preserving derivation δ, the map $\overline{\delta}$ is a derivation from R/I to R/I.
Let $\delta \in \mathrm{Ker}(\varphi)$, then $\delta(r) \in I$, for any $r \in R$. Since $\mathrm{Der}_k(R)$ is free, we can write: $\delta = \sum_i c_i \frac{\partial}{\partial x_i}$, where $c_i \in R$. Hence, $c_i = \delta(x_i) \in I$ and therefore, $\delta \in I\,\mathrm{Der}_k(R)$. For the converse direction, the same argument applies and we get: $\mathrm{Ker}(\varphi) = I\,\mathrm{Der}_k(R)$.
Now it is left to show that φ is surjective. From Proposition 3.1.11, we get an exact sequence:

$$0 \longrightarrow M \longrightarrow R/I \otimes_R \Omega^1_{R/k} \longrightarrow \Omega^1_{R/I/k} \longrightarrow 0,$$

where M denotes the image of I/I^2. If we apply $\mathrm{Hom}_R(-, R/I)$, we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Der}_k(R/I) \longrightarrow \mathrm{Der}_k(R, R/I) \xrightarrow{\beta} \mathrm{Hom}_R(M, R/I)$$

The map β sends a derivation δ to its restriction on I: $\delta_{|I}$. Note that we used:

$$\begin{aligned} \mathrm{Hom}_R(R/I \otimes_R \Omega^1_{R/k}, R/I) &= \mathrm{Hom}_R(\Omega^1_{R/k}, \mathrm{Hom}_R(R/I, R/I)) \\ &= \mathrm{Hom}_R(\Omega^1_{R/k}, R/I) \\ &= \mathrm{Der}_k(R, R/I) \end{aligned}$$

The sequence $0 \to I \to R \to R/I \to 0$ is also exact and if we apply $\mathrm{Hom}_R(\Omega^1_{R/k}, -)$, we get another exact sequence:

$$\mathrm{Der}_k(R) \xrightarrow{\phi} \mathrm{Der}_k(R, R/I) \longrightarrow \mathrm{Ext}^1_R(\Omega^1_{R/k}, I),$$

where ϕ sends a derivation δ to the derivation $\overline{\delta}$. The module $\mathrm{Ext}^1_R(\Omega^1_{R/k}, I)$ is 0, since $\Omega^1_{R/k}$ is free and thus, ϕ is surjective.

If we combine this with the above sequence, we get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{D}_I(R) \arrow[r] \arrow[d, "\varphi"] & \mathrm{Der}_k(R) \arrow[d, "\phi"] & \\
0 \arrow[r] & \mathrm{Der}_k(R/I) \arrow[r] & \mathrm{Der}_k(R, R/I) \arrow[r, "\beta"] & \mathrm{Hom}_R(M, R/I)
\end{tikzcd}$$

Let now γ be a derivation in $\mathrm{Der}_k(R/I)$. Then we can consider γ as derivation in $\mathrm{Der}_k(R, R/I)$ and $\gamma \in \mathrm{Ker}(\beta)$. Hence, $\gamma_{|I} = 0$. Since ϕ is surjective, there is a $\delta \in \mathrm{Der}_k(R)$ so that $\phi(\delta) = \gamma$. But then δ is actually I-preserving, since for $x \in I$ we have: $\overline{\delta(x)} = \phi(\delta)(x) = \gamma(x) = 0$. Therefore, $\delta(x) \in I$. This shows that φ is surjective. ■

As a consequence, we get that localization and I-preserving derivations commute:

Lemma 3.4.12. *Let k be a field and R a Noetherian k-algebra so that $\Omega^1_{R/k}$ is free of finite rank and let W be a multiplicatively closed subset of R. Then we have:*

$$W^{-1}\,\mathrm{D}_I(R) \cong \mathrm{D}_{W^{-1}I}(W^{-1}R)$$

Proof. From Lemma 3.4.11, we get the exactness of

$$0 \to I\,\mathrm{Der}_k(R) \to \mathrm{D}_I(R) \to \mathrm{Der}_k(R/I) \to 0 \tag{3.4}$$

If we localize this by W^{-1}, we can apply Proposition 3.4.2 and obtain the exact sequence:

$$0 \to W^{-1}I\,\mathrm{Der}_k(W^{-1}R) \to W^{-1}\,\mathrm{D}_I(R) \to \mathrm{Der}_k(W^{-1}R/W^{-1}I) \to 0 \tag{3.5}$$

Since $\Omega^1_{W^{-1}R/k} = W^{-1}\Omega^1_{R/k}$ by Lemma 3.1.16, we get that $\Omega^1_{W^{-1}R/k}$ is also free and thus, the following sequence is exact:

$$0 \to W^{-1}I\,\mathrm{Der}_k(W^{-1}R) \to \mathrm{D}_{W^{-1}I}(W^{-1}R) \to \mathrm{Der}_k(W^{-1}R/W^{-1}I) \to 0 \tag{3.6}$$

We have a map of exact sequences from (3.4) to (3.6). Hence, this induces a map of exact sequences from (3.5) to (3.6). By the Snake Lemma, we get the isomorphism: $W^{-1}\,\mathrm{D}_I(R) \cong \mathrm{D}_{W^{-1}}(W^{-1}R)$. ■

4. Invariants

Invariants are a good tool to state numerical characterizations of abstract objects and we are interested in computing them. We want to do this on "easy" rings: quotients of localized polynomial rings over a field k. Afterwards, we wish to extend the results to more complicated rings, like (formal) analytic algebras over L, where L/k is a field extension. Therefore, we need to show that the invariants that we treat in this chapter, behave well under completion and field extension. Since invariants are defined via the length of a module, we will show in Section 4.1, that the length behaves well with respect to mentioned operations.
Section 4.2 focuses on the delta invariant. This is the "most basic" invariant, that we consider and other invariants are related to it. In Section 4.3, we first look at a special ideal that connects the normalization $\overline{S}$ and the ground ring S: the conductor. Then we focus on the multiplicity of the conductor: an invariant, defined in terms of the conductor. The last Section, 4.4, treats the Deligne number. This is an invariant which relates derivations of S and derivations of $\overline{S}$.

4.1 The preservation of length

This short section collects results about the preservation of the length of a module under certain operations. We will show that under suitable assumptions, the length is preserved under completion and extension of the ground field. Since *invariants* are defined via the length, these results are used in the following sections to show that *invariants* are stable under the mentioned operations.

All operations we consider, are flat. Hence, it is not surprising, that the results in this section mainly rely on the following:

Proposition 4.1.1. *Let $R \to S$ be a flat local homomorphism of local rings and let $\mathfrak{m}$ be the maximal ideal of R, then for any R-module M, we have:*

$$\text{length}_S(M \otimes_R S) = \text{length}_S(S/\mathfrak{m}S) \cdot \text{length}_R(M)$$

Proof. See [Stacks, Tag 02M1]. ■

We will also need the following fact on the length of quotients by maximal ideals:

Lemma 4.1.2. *Let R be a ring and $\mathfrak{m}$ a maximal ideal of R. Then:*

$$\operatorname{length}_R(R/\mathfrak{m}) = 1$$

Proof. Any R-submodule of $R/\mathfrak{m}$ is induced by an R-submodule of R, an ideal, containing $\mathfrak{m}$. This can only be $\mathfrak{m}$ or R, hence $R/\mathfrak{m}$ only has trivial R-submodules. ■

We know from Section 1.4, that the completion of a local ring is again a local ring and that the maximal ideal maps to the maximal ideal of the completion. Therefore, we are able to apply Proposition 4.1.1 in this case:

Proposition 4.1.3. *Let R be a local Noetherian ring, $\mathfrak{q}$ an ideal of R and M a finitely generated R-module. Then $\mathfrak{q}$-adic completion preserves the length of M:*

$$\operatorname{length}_{\widehat{R}}(\widehat{M}) = \operatorname{length}_R(M)$$

Proof. The map $R \to \widehat{R}$ is a flat local homomorphism of local rings by Proposition 1.4.19 and Corollary 1.4.13. Hence, we may apply Proposition 4.1.1 and Theorem 1.4.11 to get:

$$\operatorname{length}_{\widehat{R}}(\widehat{M}) = \operatorname{length}_{\widehat{R}}(M \otimes_R \widehat{R}) = \operatorname{length}_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R}) \cdot \operatorname{length}_R(M),$$

where $\mathfrak{m}$ denotes the maximal ideal of R. Since $\mathfrak{m}\widehat{R} = \widehat{\mathfrak{m}}$ is the maximal ideal of $\widehat{R}$, the module $\widehat{R}/\mathfrak{m}\widehat{R}$ has length 1 by Lemma 4.1.2. This shows the equality. ■

We also want such a result, when we extend the ground field. But if we start with a local k-algebra S and L is the extension field of k, the L-algebra $S \otimes_k L$ may not be local anymore and we cannot apply Proposition 4.1.1. Therefore, we first extend the field and then localize at a maximal ideal of $S \otimes_k L$. We will obtain a flat local homomorphism of local rings and the length will be preserved:

Proposition 4.1.4. *Let k be a field and S a local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. If L is an extension field of k, then:*

a) The ideal $\mathfrak{n} = \mathfrak{m}(S \otimes_k L)$ is a maximal ideal of $S \otimes_k L$.

b) Set $T = (S \otimes_k L)_{\mathfrak{n}}$, then for any S-module M, we have:

$$M \otimes_S T = (M \otimes_k L)_{\mathfrak{n}}$$

c) For any S-module M, the length is preserved under tensoring by T:

$$\operatorname{length}_T(M \otimes_S T) = \operatorname{length}_S(M)$$

d) The dimension does not change if we pass to T:

$$\dim T = \dim S$$

Proof. To prove part a), we apply Lemma A.2.1:

$$(S \otimes_k L)/\mathfrak{n} = S/\mathfrak{m} \otimes_k L = k \otimes_k L = L$$

Therefore, $\mathfrak{n}$ is maximal in $S \otimes_k L$. For part b), consider:

$$\begin{aligned} M \otimes_S T &= M \otimes_S (S \otimes_k L)_{\mathfrak{n}} \\ &= M \otimes_S (S \otimes_k L) \otimes_{(S \otimes_k L)} (S \otimes_k L)_{\mathfrak{n}} \\ &= (M \otimes_k L) \otimes_{(S \otimes_k L)} (S \otimes_k L)_{\mathfrak{n}} = (M \otimes_k L)_{\mathfrak{n}} \end{aligned}$$

Finally, we prove the preservation of the length and the dimension: the map $\varphi : S \to T$ is a composition of the flat maps: $S \to (S \otimes_k L) \to (S \otimes_k L)_{\mathfrak{n}}$ and hence, flat. Applying part a), we obtain that $\varphi(\mathfrak{m}) = \mathfrak{n}_{\mathfrak{n}}$ and therefore, it is a flat local homomorphism of local rings. Now we get for an S-module M by Proposition 4.1.1:

$$\mathrm{length}_T(M \otimes_S T) = \mathrm{length}_T(T/\mathfrak{m}T) \cdot \mathrm{length}_S(M)$$

But $\mathfrak{m}T = \varphi(\mathfrak{m})T = \mathfrak{n}_{\mathfrak{n}}$ and thus, the length of $T/\mathfrak{m}T$ is 1 by Lemma 4.1.2. We also get that $\dim T = \dim S + \dim T/\mathfrak{m}T$ by Proposition A.2.5. As mentioned before, $\mathfrak{m}T$ is the maximal ideal of T and thus, $\dim T/\mathfrak{m}T = 0$. ■

4.2 Delta invariant

The first *invariant* we consider, is the *delta invariant*, denoted by δ. It measures, how "far away" a curve S is from its normalization $\overline{S}$: The *delta invariant* has value 0 if and only if S is integrally closed. In the case of curve singularities, it is also closely connected to the *Milnor number* μ by the formula $\mu = 2\delta - r + 1$, where r denotes the number of branches of the curve singularity. For the proof, we refer to [BG80, Proposition 1.2.1]. In Section 4.3, we will also see, how δ is related to the *multiplicity of the conductor*.
In this section, we will give the definition of the *delta invariant* and we will show that it is preserved under completion and extension of the ground field.

Definition 4.2.1. Let S be a reduced Noetherian ring of dimension 1 with normalization $\overline{S}$. Then the **delta invariant** of S is defined by:

$$\delta_S = \mathrm{length}_S(\overline{S}/S)$$

By definition, it is clear that $\delta_S = 0$ if and only if $S = \overline{S}$.

For illustration, let us look at an example. Therefore, we will need an exact sequence: Let R be a ring and $I = I_1 \cap I_2$ an intersection of ideals, then:

$$0 \longrightarrow R/I \longrightarrow R/I_1 \times R/I_2 \longrightarrow R/(I_1 + I_2) \longrightarrow 0$$

is exact. This will be helpful when computing the delta invariant:

Example 4.2.2. Let k be a field and $R = k[x,y]/\langle xy \rangle$. Then R has two minimal primes: $P_1 = \langle \overline{x} \rangle$ and $P_2 = \langle \overline{y} \rangle$, by Example 1.1.10. We have also seen, that $\overline{R} = R/P_1 \times R/P_2$ in Example 2.1.16. Hence, we can apply the above exact sequence and use that $P_1 \cap P_2 = 0$, since R is reduced:

$$0 \longrightarrow R \longrightarrow \overline{R} \longrightarrow R/(P_1 + P_2) \longrightarrow 0$$

So we obtain: $\overline{R}/R \cong R/(P_1 + P_2) = R/\langle \overline{x}, \overline{y} \rangle$. Since $\langle \overline{x}, \overline{y} \rangle$ is a maximal ideal of R, we may apply Lemma 4.1.2 and derive for the delta invariant:

$$\delta_R = \operatorname{length}_R(\overline{R}/R) = \operatorname{length}_R(R/\langle \overline{x}, \overline{y} \rangle) = 1$$

Now we prove that the delta invariant is compatible with completion. Because of this, we can, for example, compute the delta invariant of a formal analytic algebra over a local algebra essentially of finite type instead.

Proposition 4.2.3. *If S is a reduced excellent local ring of dimension 1 and $\mathfrak{m}$ the maximal ideal of S, then the delta invariant is stable under $\mathfrak{m}$-adic completion:*

$$\delta_{\widehat{S}} = \delta_S$$

Proof. Since S is reduced and excellent, Theorem 2.4.14 states, that S is normalization-finite and we can apply Corollary 1.4.10 to obtain: $\widehat{(\overline{S}/S)} \cong \widehat{\overline{S}}/\widehat{S}$. Another application of Theorem 2.4.14 yields: $\widehat{(\overline{S}/S)} \cong \overline{\widehat{S}}/\widehat{S}$. Hence, we get the equality by 4.1.3:

$$\delta_{\widehat{S}} = \operatorname{length}_{\widehat{S}}(\overline{\widehat{S}}/\widehat{S}) = \operatorname{length}_{\widehat{S}}(\widehat{\overline{S}/S}) = \operatorname{length}_S(\overline{S}/S) = \delta_S$$

Note that $\widehat{S}$ is reduced by Theorem 2.4.14 and of dimension 1 by Proposition 1.4.20. ∎

When we deal with algebras over fields and extend the ground field, then, under suitable assumptions, we get a similar statement to 4.2.3, as long as the ground field is perfect:

Proposition 4.2.4. *Let S be a reduced excellent local algebra of dimension 1 over a perfect field k with maximal ideal $\mathfrak{m}$, so that $S/\mathfrak{m} = k$. If L/k is a field extension, set $\mathfrak{n} = \mathfrak{m}(S \otimes_k L)$ and $T = (S \otimes_k L)_{\mathfrak{n}}$. Suppose, that $S \otimes_k L$ is Noetherian, then the delta invariant is stable under passing to T:*

$$\delta_T = \delta_S$$

Proof. First, note that $\mathfrak{n}$ is a maximal ideal of $S \otimes_k L$ by Proposition 4.1.4. The L-algebra $S \otimes_k L$ is reduced by Lemma 1.3.4, since L/k is separable. As a localization, also T is reduced and $\dim T = \dim S = 1$ by Proposition 4.1.4. Now consider the normalization of T and apply Proposition 2.1.17, Theorem 2.5.4 and Proposition 4.1.4:

$$\overline{T} = \overline{(S \otimes_k L)_{\mathfrak{n}}} = (\overline{S \otimes_k L})_{\mathfrak{n}} = (\overline{S} \otimes_k L)_{\mathfrak{n}} = \overline{S} \otimes_S T$$

Hence, we can deduce, using the flatness of T over S, Lemma A.2.1 and Proposition 4.1.4:

$$\begin{aligned}\delta_T &= \operatorname{length}_T(\overline{T}/T)\\ &= \operatorname{length}_T(\overline{S} \otimes_S T / S \otimes_S T)\\ &= \operatorname{length}_T(\overline{S}/S \otimes_S T)\\ &= \operatorname{length}_S(\overline{S}/S)\\ &= \delta_S\end{aligned}$$

■

Remark 4.2.5. We mention shortly, which kind of rings satisfy the assumptions of Proposition 4.2.4. In particular, we need that $S \otimes_k L$ is Noetherian. Therefore, let k denote a perfect field.

- Let S be a local algebra, essentially of finite type of dimension 1, satisfying $S/\mathfrak{m} = k$. Then S is excellent by Remark 2.4.13. The L-algebra $S \otimes_k L$ is again an algebra essentially of finite type by Lemma A.4.1. Hence, $S \otimes_k L$ is Noetherian.
  Examples for algebras like S are given by quotients of:

$$k[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$$

4.3 Conductor

The *conductor* of a reduced ring S, denoted $\mathfrak{C}_S$, is the module quotient of S by $\overline{S}$. We will see that it is an ideal of S and it combines all elements that multiply $\overline{S}$ "into" S. The *conductor* can also be seen as ideal of the normalization and it is even maximal with the property of being an ideal in S and $\overline{S}$. We will show that natural operations like localization, completion and extension of the ground field, do not harm $\mathfrak{C}_S$: it commutes with these. After this, we will define the *multiplicity of the conductor*, a new invariant, that is related to the delta invariant and that will also behave well with respect to completion and extension of the ground field.
Since we want to compute the *conductor* and the *multiplicity of the conductor* in Chapter 5, we give an alternative form of $\mathfrak{C}_S$ at the end of this section.

Definition 4.3.1. Let S be a reduced ring with normalization $\overline{S}$. Then the **conductor** of S, denoted by $\mathfrak{C}_S$, is the module quotient:

$$\mathfrak{C}_S = S :_{Q(S)} \overline{S} = \{x \in Q(S) \,|\, x\overline{S} \subseteq S\}$$

Remark 4.3.2. Let S be a reduced ring and $\mathfrak{C}_S$ the conductor.

- We know that $\mathfrak{C}_S\overline{S} \subseteq S$, therefore $\mathfrak{C}_S = \mathfrak{C}_S \cdot 1 \subseteq S$ and thus, the conductor is an ideal of S. Now we can also derive:

  $$\mathfrak{C}_S = S :_S \overline{S} = \mathrm{Ann}_S(\overline{S}/S)$$

  and: $\mathfrak{C}_S = S$ if and only if S is integrally closed.

- One can also show, that $\mathfrak{C}_S$ is an ideal of $\overline{S}$:

  $$\overline{S}\mathfrak{C}_S\overline{S} \subseteq \mathfrak{C}_S\overline{S} \subseteq S$$

  Hence, $\overline{S}\mathfrak{C}_S \subseteq \mathfrak{C}_S$.

- The conductor is the largest common ideal of S and $\overline{S}$: if I is another ideal of S and $\overline{S}$, then we can deduce:

  $$I\overline{S} \subseteq I \subseteq S$$

  Hence, $I \subseteq \mathfrak{C}_S$.

Example 4.3.3. Let $R = k[x, y]/\langle xy\rangle$, where k denotes a field. By Example 4.2.2, we know that $\overline{R}/R \cong R/m$, where $\mathfrak{m} = \langle \overline{x}, \overline{y}\rangle$. Then we can easily compute the conductor:

$$\mathfrak{C}_R = \mathrm{Ann}_R(\overline{R}/R) = \mathrm{Ann}_R(R/\mathfrak{m}) = \mathfrak{m}$$

In particular, we have seen, that $\mathfrak{C}_R$ contains non-zero divisors. This is a consequence of R being normalization-finite in the example.

Lemma 4.3.4. *Let S be a reduced Noetherian ring and $\mathfrak{C}_S$ its conductor. Then $\mathfrak{C}_S$ contains a non-zero divisor if and only if S is normalization-finite.*

Proof. See [HS06, p. 238]. ∎

Another useful description of the conductor that we will need later, can be derived from the following lemma:

Lemma 4.3.5. *Assume that S is a reduced ring and I, J are S-submodules of $Q(S)$, so that I contains a non-zero divisor g of S. Then the map*

$$\begin{aligned} \mathrm{Hom}_S(I, J) &\longrightarrow J :_{Q(S)} I \\ \varphi &\longmapsto \frac{\varphi(g)}{g} \end{aligned}$$

is independent of the choice of g and an isomorphism of S-modules.

Proof. This is [GLS10, Lemma 3.1]. ∎

Corollary 4.3.6. *Let S be a reduced ring and $\mathfrak{C}_S$ its conductor. Then we have an isomorphism:*

$$\operatorname{Hom}_S(\overline{S}, S) \cong \mathfrak{C}_S$$

Proof. This is an application of Lemma 4.3.5. ∎

We shall now start to examine the behaviour of the conductor with respect to localization, completion and extension of the ground field:

Proposition 4.3.7. *Let S be a reduced normalization-finite Noetherian ring and W a multiplicatively closed subset of S. Then the conductor is stable under localization:*

$$\mathfrak{C}_{W^{-1}S} \cong W^{-1}\mathfrak{C}_S$$

Proof. This is a consequence of Proposition 2.1.17, Proposition A.2.2 and Corollary 4.3.6:

$$\begin{aligned}\mathfrak{C}_{W^{-1}S} &\cong \operatorname{Hom}_{W^{-1}S}(\overline{W^{-1}S}, W^{-1}S)\\ &= \operatorname{Hom}_{W^{-1}S}(W^{-1}\overline{S}, W^{-1}S)\\ &\cong W^{-1}\operatorname{Hom}_S(\overline{S}, S)\\ &\cong W^{-1}\mathfrak{C}_S\end{aligned}$$ ∎

A similar argumentation shows, that the conductor also commutes with completion. For this, we must assume that we have an excellent ring, since we want to swap completion and normalization.

Proposition 4.3.8. *Let S be a reduced excellent semi-local ring and $\mathfrak{m}$ the Jacobson radical of S. If $\widehat{\ }$ denotes the $\mathfrak{m}$-adic completion, we have:*

$$\mathfrak{C}_{\widehat{S}} \cong \widehat{\mathfrak{C}_S}$$

Proof. By Theorem 2.4.14 we know, that S is normalization-finite, $\widehat{S}$ is reduced and that $\widehat{\overline{S}} = \overline{\widehat{S}}$. Hence, we may apply Theorem 1.4.11, Proposition A.2.2 and Corollary 4.3.6:

$$\begin{aligned}\mathfrak{C}_{\widehat{S}} &\cong \operatorname{Hom}_{\widehat{S}}(\overline{\widehat{S}}, \widehat{S})\\ &\cong \operatorname{Hom}_{S\otimes_S\widehat{S}}(\overline{S}\otimes_S \widehat{S}, S\otimes_S \widehat{S})\\ &\cong \operatorname{Hom}_S(\overline{S}, S)\otimes_S \widehat{S}\\ &\cong \widehat{\mathfrak{C}_S}\end{aligned}$$ ∎

The last operation, which is compatible with the conductor, is the extension of the ground field. The idea of the proof is similar to the above propositions.

Proposition 4.3.9. *Let S be a reduced excellent k-algebra, where k denotes a perfect field. Assume L/k is a field extension, then we have an isomorphism:*

$$\mathfrak{C}_{S \otimes_k L} \cong \mathfrak{C}_S \otimes_k L$$

Proof. The ring S is reduced and k is perfect. Hence, we get by Lemma 1.3.4 that $S \otimes_k L$ is reduced. Applying Theorem 2.5.4 and Corollary A.2.4, we can deduce:

$$\begin{aligned}\mathfrak{C}_{S \otimes_k L} &\cong \mathrm{Hom}_{S \otimes_k L}(\overline{S \otimes_k L}, S \otimes_k L)\\ &\cong \mathrm{Hom}_{S \otimes_k L}(\overline{S} \otimes_k L, S \otimes_k L)\\ &\cong \mathrm{Hom}_S(\overline{S}, S) \otimes_k L\\ &\cong \mathfrak{C}_S \otimes_k L\end{aligned}$$

∎

Now we focus on a new invariant, that was mentioned at the beginning of this section and that is defined in terms of the conductor:

Definition 4.3.10. Let S be a reduced Noetherian ring of dimension 1 and $\mathfrak{C}_S$ the conductor of S. The length of $\overline{S}/\mathfrak{C}_S$ as S-module is called **multiplicity of the conductor** and is denoted by c_S.

There is a simple connection to the delta invariant:

Remark 4.3.11. Let S be as above, $\mathfrak{C}_S$ the conductor and c_S the multiplicity of the conductor. Then there is an isomorphism of S-modules:

$$\overline{S}/S \cong (\overline{S}/\mathfrak{C}_S)\big/(S/\mathfrak{C}_S)$$

Hence, we obtain:

$$\begin{aligned}c_S &= \mathrm{length}_S(\overline{S}/\mathfrak{C}_S)\\ &= \mathrm{length}_S(\overline{S}/S) + \mathrm{length}_S(S/\mathfrak{C}_S)\\ &= \delta_S + \mathrm{length}_S(S/\mathfrak{C}_S)\end{aligned}$$

If S is also local, then there are more relations to other invariants and one can even derive a *Gorenstein* test: The ring S is *Gorenstein* if and only if, we have an equality: $c_S = 2 \cdot \mathrm{length}_S(S/\mathfrak{C}_S)$. For details we refer to [HS06, Section 12.2].

We will now show, that the multiplicity of the conductor behaves well under completion and field extension. Most of the work has already been done, it was shown that the conductor behaves well with respect to this operations.

Proposition 4.3.12. *For a reduced excellent local ring S of dimension 1, let $\mathfrak{m}$ denote the maximal ideal. The conductor multiplicity is stable under $\mathfrak{m}$-adic completion:*

$$c_{\widehat{S}} = c_S$$

Proof. First, note that $\widehat{S}$ is reduced and of dimension 1. This is a consequence of Proposition 1.4.20 and Theorem 2.4.14.
Due to the assumptions on S, we may apply Theorem 2.4.14, Proposition 4.3.8 and Proposition 1.4.14 to obtain:

$$\overline{\widehat{S}}/\mathfrak{C}_{\widehat{S}} \cong \widehat{\overline{S}}/\widehat{\mathfrak{C}_S} = \widehat{(\overline{S}/\mathfrak{C}_S)}$$

Since S is normalization-finite, we get by Proposition 4.1.3 that:

$$c_S = \operatorname{length}_S(\overline{S}/\mathfrak{C}_S) = \operatorname{length}_{\widehat{S}}(\overline{\widehat{S}}/\mathfrak{C}_{\widehat{S}}) = c_{\widehat{S}}$$

■

Proposition 4.3.13. *Let S be a reduced excellent local algebra of dimension 1 over a perfect field k with maximal ideal $\mathfrak{m}$, so that $S/\mathfrak{m} = k$. If L/k is a field extension, set $\mathfrak{n} = \mathfrak{m}(S \otimes_k L)$ and $T = (S \otimes_k L)_\mathfrak{n}$. Suppose, that $S \otimes_k L$ is Noetherian, then the multiplicity of the conductor is stable under passing to T:*

$$c_T = c_S$$

Proof. Like in Proposition 4.2.4, we can derive that T is reduced, $\dim T = 1$ and $\overline{T} = \overline{S} \otimes_S T$. The conductor behaves the same way:

$$\mathfrak{C}_T = \mathfrak{C}_{(S \otimes_k L)_\mathfrak{n}} \cong (\mathfrak{C}_{S \otimes_k L})_\mathfrak{n} \cong (\mathfrak{C}_S \otimes_k L)_\mathfrak{n} = \mathfrak{C}_S \otimes_S T$$

This is a consequence of Propositions 4.3.7, 4.3.9 and 4.1.4. Note, that $S \otimes_k L$ is reduced by Lemma 1.3.4 and that $S \otimes_k L$ is normalization-finite: S is normalization-finite and by Theorem 2.5.4, we can write $\overline{S \otimes_k L} = \overline{S} \otimes_k L$, which is finitely generated over $S \otimes_k L$.
Now we can derive, using Lemma A.2.1 and again Proposition 4.1.4:

$$\begin{aligned} c_T &= \operatorname{length}_T(\overline{T}/\mathfrak{C}_T) \\ &= \operatorname{length}_T(\overline{S} \otimes_S T/\mathfrak{C}_S \otimes_S T) \\ &= \operatorname{length}_T(\overline{S}/\mathfrak{C}_S \otimes_S T) \\ &= \operatorname{length}_S(\overline{S}/\mathfrak{C}_S) \\ &= c_S \end{aligned}$$

■

Our last consideration in this section aims for finding a form for the conductor of a reduced Noetherian normalization-finite ring S, which depends on the minimal associated primes and the S-module generators of $\overline{S}$. In the proof, we will make use of the following fact:

Lemma 4.3.14. *Let R be a ring, M an arbitrary R-module and N a finitely generated R-module with generators: $n_1, \ldots, n_k$. Then:*

$$M :_R N = \bigcap_{i=1}^{k} (M :_R n_i)$$

Proof. Let $r \in R$. Then $rN \subseteq M$ if and only if $rn_i \in M$ for any generator of N. ■

Definition 4.3.15. Let S be a reduced, Noetherian ring with minimal primes $P_1, \dots, P_r$. The map $\Delta : S \to S/P_1 \times \cdots \times S/P_r$, defined by $\Delta(x) = (\overline{x}, \dots, \overline{x})$ is called **diagonal embedding**.

Since S is reduced and the intersection of all minimal primes is the nilradical by Proposition 1.2.2, Δ is indeed an injective ring homomorphism:

$$\operatorname{Ker}(\Delta) = \bigcap_{i=1}^{r} P_i = \mathfrak{N}(S) = 0$$

With Proposition 2.1.15 and Δ, we can identify S as subring of the normalization:

$$S \cong \Delta(S) \subseteq S/P_1 \times \cdots \times S/P_r \subseteq \overline{S/P_1} \times \cdots \times \overline{S/P_r} = \overline{S}$$

This allows us to "test" whether an element of $\overline{S}$ already lies in S in the next theorem.

Theorem 4.3.16. *Let S be a reduced Noetherian ring with minimal primes $P_1, \dots, P_r$. Suppose, we have $U_1, \dots, U_r \trianglelefteq S$ and $d_1, \dots, d_r \in S$ such that $\overline{d_i} \neq 0$ in S/P_i and $\overline{S/P_i} = \frac{1}{d_i} U_i \subseteq Q(S/P_i)$ as an S-module. Also assume, that there are presentations as S-algebras: $\overline{S/P_i} = S_i/J_i$, where $S_i = S[\underline{t_i}]$ and $J_i \trianglelefteq S_i$.*
Then the conductor $\mathfrak{C}_S$ of S is the ideal:

$$\bigcap_{i=1}^{r} \left(\left(P_i + d_i \bigcap_{j \neq i} P_j \right) :_S U_i \right) \trianglelefteq S.$$

Proof. First, we show that $J_i \cap S = P_i$ for all i. Consider the commutative diagram:

$$\begin{tikzcd} S \arrow[d, "\pi"] \arrow[dr, "\varphi_i"] & \\ S/P_i \arrow[r, hook] & \overline{S/P_i} = S_i/J_i \end{tikzcd}$$

where φ_i is the map: $\varphi_i : S \to S_i/J_i$, $x \mapsto \overline{x}$. From this, we obtain:

$$J_i \cap S = \operatorname{Ker}(\varphi_i) = \operatorname{Ker}(\pi) = P_i.$$

Now we take a look at generators of $\overline{S}$ as an S-module. Therefore, let $U_i = \langle u_{i_l}\ ,\ l = 1, \dots, k_i \rangle$, then by assumption $\overline{S/P_i} = \frac{1}{d_i} U_i = \left\langle \frac{\overline{u_{i_l}}}{d_i}\ ,\ l = 1, \dots, k_i \right\rangle_S$. Hence, the S-module $\overline{S} = \overline{S/P_1} \times \cdots \times \overline{S/P_r}$ is generated by

$$\left(0, \dots, 0, \frac{\overline{u_{i_l}}}{d_i}, 0, \dots, 0\right),\ i = 1, \dots, r,\ l = 1, \dots, k_i$$

Applying Lemma 4.3.14, we can conclude for the conductor:

$$\begin{aligned}
\mathfrak{C}_S &= S :_S \overline{S} = \Delta(S) :_S \overline{S} \\
&= \bigcap_{i=1}^{r} \bigcap_{l=1}^{k_i} \Delta(S) :_S \left\langle \left(0, \ldots, 0, \frac{\overline{u_{i_l}}}{d_i}, 0, \ldots, 0\right) \right\rangle_S
\end{aligned} \tag{4.1}$$

Now focus on the quotient of a single generator. Therefore let $f \in S$, then:

$$\begin{aligned}
& f \in \Delta(S) :_S \left\langle \left(0, \ldots, 0, \frac{\overline{u_{i_l}}}{d_i}, 0, \ldots, 0\right) \right\rangle_S \\
\Leftrightarrow & f \cdot \left(0, \ldots, 0, \frac{\overline{u_{i_l}}}{d_i}, 0, \ldots, 0\right) \in \Delta(S) \\
\Leftrightarrow & \exists\ g \in S\colon \left(0, \ldots, 0, \overline{f\frac{u_{i_l}}{d_i}}, 0, \ldots, 0\right) = \Delta(g) \\
\Leftrightarrow & \exists\ g \in S\colon \begin{cases} \overline{g} = \overline{0} \text{ in } \overline{S/P_j} = S_j/J_j \text{ for } j \neq i \text{ and} \\ \overline{g} = \overline{f\frac{u_{i_l}}{d_i}} \text{ in } \overline{S/P_i} = S_i/J_i \end{cases}
\end{aligned}$$

Since $\overline{d_i}$ is non-zero in S/P_i, we have an equivalence:

$$\overline{g} = \overline{f\frac{u_{i_l}}{d_i}} \in Q(S/P_i) \Leftrightarrow \overline{d_i g} = \overline{f u_{i_l}} \in S/P_i$$

Hence, we can continue the equivalence and make use of what we have shown first:

$$\begin{aligned}
\Leftrightarrow & \exists\ g \in S\colon \begin{cases} g \in J_j \cap S \text{ for } j \neq i \text{ and} \\ f u_{i_l} - d_i g \in J_i \cap S \end{cases} \\
\Leftrightarrow & \exists\ g \in S\colon \begin{cases} g \in P_j \text{ for } j \neq i \text{ and} \\ f u_{i_l} - d_i g \in P_i \end{cases} \\
\Leftrightarrow & \exists\ g \in S\colon g \in \bigcap_{j \neq i} P_j \text{ and } f u_{i_l} - d_i g \in P_i \\
\Leftrightarrow & f u_{i_l} \in P_i + d_i \bigcap_{j \neq i} P_j \\
\Leftrightarrow & f \in \left(P_i + d_i \bigcap_{j \neq i} P_j\right) :_S u_{i_l}
\end{aligned}$$

Now combine (4.1), the equivalences and Lemma 4.3.14 to obtain the desired form of the conductor:

$$\mathfrak{C}_S = \bigcap_{i=1}^{r} \bigcap_{l=1}^{k_i} \left(\left(P_i + d_i \bigcap_{j \neq i} P_j \right) :_S u_{i_l} \right) = \bigcap_{i=1}^{r} \left(\left(P_i + d_i \bigcap_{j \neq i} P_j \right) :_S U_i \right)$$

■

4.4 The Deligne number

Before we can introduce the *Deligne number*, we have to extend derivations from an k-algebra S to its normalization $\overline{S}$. Therefore, we will use a famous result of Seidenberg, which allows us to do so if S is an integral domain. To achieve this also in the reduced case, we first embed $\mathrm{Der}_k(S)$ into the product $\prod \mathrm{Der}_k(S/P)$, where P runs over all minimal primes of S. After this, we extend elements of this product componentwise to $\prod \mathrm{Der}_k(\overline{S/P}) = \mathrm{Der}_k(\overline{S})$, so that we have an injection $\mathrm{Der}_k(S) \hookrightarrow \mathrm{Der}_k(\overline{S})$. The *Deligne number* will then be defined in terms of the length of $\mathrm{Der}_k(\overline{S})/\,\mathrm{Der}_k(S)$.
Like for the other invariants, we will show that the *Deligne number* is stable under completion and field extension.
The second part of the section deals with the case, in which S is an algebra, essentially of finite type over a field k. We will derive a formula, that allows us to compute the *Deligne number* in Chapter 5.

As mentioned above, the first step to extend derivations from $\mathrm{Der}_k(S)$ to $\mathrm{Der}_k(\overline{S})$ is an injection $\mathrm{Der}_k(S) \hookrightarrow \prod \mathrm{Der}_k(S/P)$. Such a map is generated by the following lemma.

Lemma 4.4.1. *Let k be a field and S a reduced k-algebra with finitely many minimal primes: $P_1, \dots, P_r$. Let $I = \bigcap_{i=1}^{r-1} P_i$, then we have an injective map:*

$$\begin{aligned} \varphi : \mathrm{Der}_k(S) &\longrightarrow \mathrm{Der}_k(S/I) \times \mathrm{Der}_k(S/P_r) \\ \delta &\longmapsto (\overline{\delta}_I, \overline{\delta}_{P_r}) \end{aligned}$$

Proof. The map φ is well defined: let $\delta \in \mathrm{Der}_k(S)$, then we may apply Proposition 3.4.10 and obtain, that $\delta(P_i) \subseteq P_i$ for any i. As a consequence, $\delta(I) \subseteq I$ and $\delta(P_r) \subseteq P_r$. Hence, the maps:

$$\begin{aligned} \overline{\delta}_I : S/I &\to S/I \qquad & \overline{\delta}_{P_r} : S/P_r &\to S/P_r \\ \overline{x} &\to \overline{\delta(x)} & \overline{x} &\to \overline{\delta(x)} \end{aligned}$$

are well defined derivations.
For showing, that φ is injective, let $\delta \in \mathrm{Ker}(\varphi)$. Then we have for $x \in S$: $\delta(x) \in I \cap P_r = \bigcap_{i=1}^{r} P_i = \mathfrak{N}(S) = 0$, since S is reduced. Hence, δ is the zero map and φ is injective. ■

Remark 4.4.2. Note, that the ring S/I is again reduced, since $\sqrt{I} = \bigcap_{i=1}^{r-1} \sqrt{P_i} = \bigcap_{i=1}^{r-1} P_i = I$ and S/I has the minimal primes $P_1, \dots, P_{r-1}$ by Lemma 1.1.9. We can therefore apply Lemma 4.4.1 inductively, which yields a sequence of injective maps:

$$\mathrm{Der}_k(S) \hookrightarrow \mathrm{Der}_k(S/\bigcap_{i=1}^{r-1} P_i) \times \mathrm{Der}_k(S/P_r) \hookrightarrow \dots \hookrightarrow \prod_{i=1}^{r} \mathrm{Der}_k(S/P_i)$$

If we compose all these maps, we get an embedding:

$$\begin{aligned} \Delta^D : \mathrm{Der}_k(S) &\longrightarrow \prod_{i=1}^{r} \mathrm{Der}_k(S/P_i) \\ \delta &\longmapsto (\overline{\delta}_{P_1}, \ldots, \overline{\delta}_{P_r}) \end{aligned}$$

Inspired by the *diagonal submodule* that is generated by the image of Δ^D, we call the map **diagonal D-embedding**.

The second step to extend derivations to the normalization relies on a result of Seidenberg. We have to focus on algebras of characteristic 0 now:

Theorem 4.4.3. *Let k be a field of characteristic 0 and S an k-algebra. Suppose, S is a Noetherian integral domain with quotient field K and $\delta : K \to K$ is an k-derivation. If $\delta(S) \subseteq S$, then we have: $\delta(\overline{S}) \subseteq \overline{S}$.*

Proof. The proof is a combination of [Sei66, p. 171] and [Sei66, Theorem C]. ■

We can now combine this result and the above construction of the diagonal D-embedding to obtain the injection, we were looking for:

Proposition 4.4.4. *Let k be a field of characteristic 0 and S a reduced Noetherian k-algebra. Then, there is an embedding:*

$$\mathrm{Der}_k(S) \hookrightarrow \mathrm{Der}_k(\overline{S})$$

So we can uniquely extend any derivation from S to a derivation of $\overline{S}$.

Proof. Let P be a minimal associated prime of S and $\gamma \in \mathrm{Der}_k(S/P)$. We can extend γ uniquely to a derivation $\gamma^e \in \mathrm{Der}_k(Q(S/P))$ by Lemma 3.1.3. Since S/P is a Noetherian integral domain over k, we may apply Theorem 4.4.3 and obtain that $\gamma^e(\overline{S/P}) \subseteq \overline{S/P}$. Hence, $\gamma^e \in \mathrm{Der}_k(\overline{S/P})$. The extension γ^e is unique, therefore we get an injective map $\mathrm{Der}_k(S/P) \hookrightarrow \mathrm{Der}_k(\overline{S/P})$. Now we do this in every component and obtain an embedding:

$$\Phi : \prod_{P \in \mathrm{Min}(S)} \mathrm{Der}_k(S/P) \hookrightarrow \prod_{P \in \mathrm{Min}(S)} \mathrm{Der}_k(\overline{S/P}) = \mathrm{Der}_k(\overline{S})$$

Note, that the equality $\prod \mathrm{Der}_k(\overline{S/P}) = \mathrm{Der}_k(\overline{S})$ is a consequence of Proposition 3.4.7 and Proposition 2.1.15. We obtain the desired injection by composing Δ^D and Φ. ■

Now we are able to define the invariant:

Definition 4.4.5. Let k be a field of characteristic 0 and S a reduced Noetherian k-algebra of dimension 1.

a) Call $m_S = \mathrm{length}_S(\mathrm{Der}_k(\overline{S})/\,\mathrm{Der}_k(S))$ the **colength of derivations (along the normalization)** of S.

b) The **Deligne number** e_S is defined by: $e_S = 3\delta_S - m_S$, where δ_S denotes the delta invariant from Section 4.2.

Before we focus on the computation of the Deligne number, we prove that it is stable under completion and extension of the ground field.

Theorem 4.4.6. *Let k be a field of characteristic 0 and S a reduced excellent local algebra of dimension 1 over k so that $\widetilde{\Omega}^1_{S/k}$ exists. Denote by $\mathfrak{m}$ the maximal ideal of S, then the Deligne number is stable under $\mathfrak{m}$-adic completion:*

$$e_{\widehat{S}} = e_S$$

Proof. First of all, we may apply Proposition 4.2.3 and get that the delta invariant δ_S of S is stable under $\mathfrak{m}$-adic completion. Since $e_S = 3\delta_S - m_S$, we only have to show that the colength of derivations is stable.

Since the universally finite module of differentials $\widetilde{\Omega}^1_{S/k}$ exists by assumption, we can apply Proposition 3.4.5 to get:

$$\mathrm{Der}_k(\widehat{S}) \cong \widehat{\mathrm{Der}_k(S)}$$

The algebra S is normalization-finite by Theorem 2.4.14. Hence, $\overline{S}$ is a semi-local Noetherian ring by Proposition 2.2.8 and Lemma 2.2.7. Applying Corollary 3.3.14, we also get the existence of $\widetilde{\Omega}^1_{\overline{S}/k}$. Now denote by $\mathfrak{n}$ the Jacobson radical of $\overline{S}$. By Proposition 3.4.5, we get for the $\mathfrak{n}$-adic completion:

$$\mathrm{Der}_k(\widehat{\overline{S}}^{\mathfrak{n}}) \cong \widehat{\mathrm{Der}_k(\overline{S})}^{\mathfrak{n}}$$

Since the $\mathfrak{n}$-adic completion and the $\mathfrak{m}$-adic completion of $\overline{S}$-modules coincide by Proposition 2.2.9, we get the equalities:

$$\widehat{\overline{S}}^{\mathfrak{n}} = \widehat{\overline{S}},\ \ \mathrm{Der}_k(\widehat{\overline{S}}^{\mathfrak{n}}) = \mathrm{Der}_k(\widehat{\overline{S}}) \text{ and } \widehat{\mathrm{Der}_k(\overline{S})}^{\mathfrak{n}} = \widehat{\mathrm{Der}_k(\overline{S})}$$

Thus, we have the isomorphism

$$\mathrm{Der}_k(\,\widehat{\overline{S}}\,) \cong \widehat{\mathrm{Der}_k(\overline{S})}$$

Note, that the derivation module $\mathrm{Der}_k(\overline{S})$ is a finitely generated $\overline{S}$-module, since it is the dual of $\widetilde{\Omega}^1_{\overline{S}/k}$. The map $S \to \overline{S}$ is finite. Hence, $\mathrm{Der}_k(\overline{S})$ is also a finitely generated S-module.

Now we combine these results with Theorem 2.4.14, Corollary 1.4.10 and Proposition 4.1.3:

$$\begin{aligned} m_{\widehat{S}} &= \operatorname{length}_{\widehat{S}}(\operatorname{Der}_k(\overline{\widehat{S}})/\operatorname{Der}_k(\widehat{S})) \\ &= \operatorname{length}_{\widehat{S}}(\operatorname{Der}_k(\widehat{\overline{S}})/\operatorname{Der}_k(\widehat{S})) \\ &= \operatorname{length}_{\widehat{S}}(\widehat{\operatorname{Der}_k(\overline{S})}/\widehat{\operatorname{Der}_k(S)}) \\ &= \operatorname{length}_{\widehat{S}}((\operatorname{Der}_k(\overline{S})/\operatorname{Der}_k(S))\widehat{\ }) \\ &= \operatorname{length}_{S}(\operatorname{Der}_k(\overline{S})/\operatorname{Der}_k(S)) \\ &= m_S \end{aligned}$$

■

Note that, like in the other stability results, we have that $\widehat{S}$ is reduced and of dimension 1 and, concerning the next theorem, the ring T is reduced and of dimension 1.
For the extension of the ground field, we need the assumption that the module of Kähler differentials is finitely generated.

Theorem 4.4.7. *Let S be a reduced excellent local algebra of dimension 1 over a field k of characteristic 0. Denote by $\mathfrak{m}$ the maximal ideal of S and suppose that $S/\mathfrak{m} = k$ and that $\Omega^1_{S/k}$ is finitely generated. If L/k is a field extension, set $\mathfrak{n} = \mathfrak{m}(S \otimes_k L)$ and $T = (S \otimes_k L)_{\mathfrak{n}}$. Suppose, that $S \otimes_k L$ is Noetherian, then the Deligne number is stable under passing to T:*

$$e_T = e_S$$

Proof. The delta invariant δ_S is stable under passing to T by Proposition 4.2.4. So we focus on the colength of derivations m_S. By assumption, we know that $\Omega^1_{S/k}$ is finitely generated over S. Hence, $\Omega^1_{(S\otimes_k L)/L} = \Omega^1_{S/k} \otimes_k L$ is finitely generated over $S \otimes_k L$. The ring $S \otimes_k L$ is Noetherian by assumption, so we may apply Proposition 3.4.2 and obtain:

$$\operatorname{Der}_L(T) = \operatorname{Der}_L((S \otimes_k L)_{\mathfrak{n}}) \cong \operatorname{Der}_L(S \otimes_k L)_{\mathfrak{n}}$$

If we also apply Proposition 3.4.6 and Proposition 4.1.4, we get:

$$\operatorname{Der}_L(T) \cong (\operatorname{Der}_k(S) \otimes_k L)_{\mathfrak{n}} = \operatorname{Der}_k(S) \otimes_S T$$

Now we want to derive a similar isomorphism for $\operatorname{Der}_L(\overline{T})$: note, that the field k is perfect by Lemma 1.3.3 and since S is reduced, also $S \otimes_k L$ is reduced by Lemma 1.3.4. Hence, we are able to apply Proposition 2.1.17 to get: $\overline{(S \otimes_k L)_{\mathfrak{n}}} = (\overline{S \otimes_k L})_{\mathfrak{n}}$. Now, taking Theorem 2.5.4 into account, we can deduce:

$$\operatorname{Der}_L(\overline{T}) = \operatorname{Der}_L(\overline{(S \otimes_k L)_{\mathfrak{n}}}) = \operatorname{Der}_L((\overline{S \otimes_k L})_{\mathfrak{n}}) = \operatorname{Der}_L((\overline{S} \otimes_k L)_{\mathfrak{n}})$$

Since S is normalization-finite and $\Omega^1_{S/k}$ is finitely generated, we can use Lemma 3.1.15 and obtain that $\Omega^1_{\overline{S}/k}$ is finitely generated over $\overline{S}$. Then $\Omega^1_{(\overline{S}\otimes_k L)/L} = \Omega^1_{\overline{S}/k} \otimes_k L$ is also finitely generated - over $\overline{S}\otimes_k L$. The ring $\overline{S}\otimes_k L$ is the normalization of $S \otimes_k L$ and if we again use, that S is normalization-finite, we can derive that $S \otimes_k L$ is also normalization-finite and $\overline{S} \otimes_k L$ is Noetherian by Lemma 2.2.7. We are now able to apply the propositions 3.4.2, 3.4.6 and 4.1.4:

$$\begin{aligned}
\operatorname{Der}_L(\overline{T}) &= \operatorname{Der}_L((\overline{S} \otimes_k L)_{\mathfrak{n}}) \\
&\cong \operatorname{Der}_L(\overline{S} \otimes_k L)_{\mathfrak{n}} \\
&\cong (\operatorname{Der}_k(\overline{S}) \otimes_k L)_{\mathfrak{n}} \\
&= \operatorname{Der}_k(\overline{S}) \otimes_S T
\end{aligned}$$

Hence, we can deduce with another application of Proposition 4.1.4:

$$\begin{aligned}
m_T &= \operatorname{length}_T(\operatorname{Der}_L(\overline{T})/\operatorname{Der}_L(T)) \\
&= \operatorname{length}_T(\operatorname{Der}_k(\overline{S})/\operatorname{Der}_k(S) \otimes_S T) \\
&= \operatorname{length}_S(\operatorname{Der}_k(\overline{S})/\operatorname{Der}_k(S)) \\
&= m_S
\end{aligned}$$ ■

For later computations, we wish to reduce the extension of derivations to the single-branch case. The embedding Δ^D allows us to do so: we can state a formula, which relates the total colength of derivations m_S to the colength of derivations of the branches $m_{S/P}$ and includes the cokernel of Δ^D.

Proposition 4.4.8. *Let S be a reduced Noetherian algebra of dimension* 1 *over a field k of characteristic* 0*, then we have the formula:*

$$m_S = \sum_{P \in \operatorname{Min}(S)} m_{S/P} + \operatorname{length}_S(\operatorname{Coker}(\Delta^D))$$

Proof. The embedding $\operatorname{Der}_k(S) \hookrightarrow \operatorname{Der}_k(\overline{S})$ was defined as composition of two injective maps in Proposition 4.4.4:

$$\operatorname{Der}_k(S) \xrightarrow{\Delta^D} \prod_{P \in \operatorname{Min}(S)} \operatorname{Der}_k(S/P) \xrightarrow{\Phi} \operatorname{Der}_k(\overline{S})$$

Hence, we can deduce for the colength of derivations:

$$\begin{aligned}
m_S = {} & \operatorname{length}_S(\operatorname{Der}_k(\overline{S}) \Big/ \prod_{P \in \operatorname{Min}(S)} \operatorname{Der}_k(S/P)) \\
& + \operatorname{length}_S(\prod_{P \in \operatorname{Min}(S)} \operatorname{Der}_k(S/P) \Big/ \operatorname{Der}_k(S))
\end{aligned} \tag{4.2}$$

Taking Proposition 3.4.7 into account, we can use that the derivation module $\operatorname{Der}_k(\overline{S})$ factors:

$$\begin{aligned}\operatorname{Der}_k(\overline{S}) \Big/ \prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(S/P) &= \prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(\overline{S/P}) \Big/ \prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(S/P) \\ &\cong \prod_{P\in\operatorname{Min}(S)} (\operatorname{Der}_k(\overline{S/P})/\operatorname{Der}_k(S/P))\end{aligned}$$

Hence, we get a nice representation of the first summand of (4.2):

$$\operatorname{length}_S(\operatorname{Der}_k(\overline{S}) \Big/ \prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(S/P)) = \sum_{P\in\operatorname{Min}(S)} m_{S/P}$$

Now, we wish to find a nice representation of the second summand. Therefore, consider the exact sequence:

$$0 \longrightarrow \operatorname{Der}_k(S) \xrightarrow{\Delta^D} \prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(S/P) \longrightarrow \operatorname{Coker}(\Delta^D) \longrightarrow 0$$

Then immediately:

$$\operatorname{length}_S(\prod_{P\in\operatorname{Min}(S)} \operatorname{Der}_k(S/P) \Big/ \operatorname{Der}_k(S)) = \operatorname{length}_S(\operatorname{Coker}(\Delta^D))$$

■

We would like to compute the Deligne number over quotients of localized polynomial rings. With the stability results, we can then lift our computations to (formal) analytic algebras. To find a formula for the Deligne number and the colength of derivations, we should take a closer look at the structure of the cokernel of the diagonal D-embedding, since we have already stated a formula containing the length of $\operatorname{Coker}(\Delta^D)$ in Proposition 4.4.8.

Proposition 4.4.9. *Let S be a localization of a polynomial ring over a field k, I a radical ideal and $I = \bigcap_{i=1}^r P_i$ a decomposition into its minimal primes. Then there is an exact sequence:*

$$0 \longrightarrow \operatorname{Der}_k(S/I) \xrightarrow{\quad\varphi\quad} \operatorname{Der}_k(S/I') \times \operatorname{Der}_k(S/P_r) \xrightarrow{\quad\psi\quad} \dots$$

$$\dots \xrightarrow{\quad\psi\quad} (\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S))/(I' + P_r)\operatorname{Der}_k(S) \longrightarrow 0,$$

where $I' = \bigcap_{i=1}^{r-1} P_i$.

Proof. The map

$$\begin{aligned}\varphi : \mathrm{Der}_k(S/I) &\longrightarrow \mathrm{Der}_k(S/I') \times \mathrm{Der}_k(S/P_r)\\ \delta &\longrightarrow (\overline{\delta}_{I'}, \overline{\delta}_{P_r})\end{aligned}$$

was already constructed in Lemma 4.4.1 and shown to be injective. We may use the identification from Lemma 3.4.11 to consider φ as map:

$$\begin{aligned}\varphi : \mathrm{D}_I(S)/I\,\mathrm{Der}_k(S) &\longrightarrow \mathrm{D}_{I'}(S)/I'\,\mathrm{Der}_k(S) \times \mathrm{D}_{P_r}(S)/P_r\,\mathrm{Der}_k(S)\\ \overline{\delta} &\longmapsto (\overline{\delta}, \overline{\delta})\end{aligned}$$

Define the map ψ by:

$$\begin{aligned}\psi : \mathrm{D}_{I'}(S)/I'\,\mathrm{Der}_k(S) \times \mathrm{D}_{P_r}(S)/P_r\,\mathrm{Der}_k(S) &\longrightarrow (\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S))/(I' + P_r)\,\mathrm{Der}_k(S)\\ (\overline{\delta}, \overline{\epsilon}) &\longmapsto \overline{\delta - \epsilon}\end{aligned}$$

Then:

- *ψ is well defined:* For $\delta \in \mathrm{D}_{I'}(S)$ and $\epsilon \in \mathrm{D}_{P_r}(S)$, it is clear that $\delta - \epsilon$ lies in $\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S)$.

  Now let $\delta = \hat{\delta} + \delta'$, where δ and $\hat{\delta}$ represent the same residue class in $\mathrm{D}_{I'}(S)/I'\,\mathrm{Der}_k(S)$ and $\delta' \in I'\,\mathrm{Der}_k(S) \subseteq (I' + P_r)\,\mathrm{Der}_k(S)$. Similarly, $\epsilon = \hat{\epsilon} + \epsilon'$ with $\epsilon' \in P_r\,\mathrm{Der}_k(S) \subseteq (I' + P_r)\,\mathrm{Der}_k(S)$. Then, $\delta - \epsilon = \hat{\delta} - \hat{\epsilon} + \delta' - \epsilon'$ and $\delta' - \epsilon' \in (I' + P_r)\,\mathrm{Der}_k(S)$, so $\delta - \epsilon$ and $\hat{\delta} - \hat{\epsilon}$ represent the same class in $(\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S))/(I' + P_r)\,\mathrm{Der}_k(S)$.

- *ψ is surjective:* Let $\overline{\delta + \epsilon} \in (\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S))/(I' + P_r)\,\mathrm{Der}_k(S)$. Then $\delta \in \mathrm{D}_{I'}(S)$ and $\epsilon \in \mathrm{D}_{P_r}(S)$. Hence, $\overline{\delta + \epsilon}$ is the image of $(\overline{\delta}, \overline{-\epsilon})$.

To get the exactness of the sequence, we only need to prove that $\mathrm{Ker}(\psi) \subseteq \mathrm{Im}(\varphi)$. The other inclusion is clear by definition of the maps.

So let $(\overline{\delta}, \overline{\epsilon}) \in \mathrm{Ker}(\psi)$. It follows that $\delta - \epsilon \in (I' + P_r)\,\mathrm{Der}_k(S)$ and thus, there exist $a \in I'$, $b \in P_r$ and $\delta' \in \mathrm{Der}_k(S)$ such that:

$$\delta - \epsilon = (a + b)\delta'$$

By resorting the equation, we obtain:

$$\delta - a\delta' = \epsilon + b\delta'$$

Set $\gamma = \delta - a\delta'$, then we get for $x \in I$:

$$\begin{aligned}\gamma(x) &= \underbrace{\delta(x)}_{\in I',\text{ since }\delta \in \mathrm{D}_{I'}(S)} - \underbrace{a\delta'(x)}_{\in I',\text{ since }a \in I'} \in I'\\ \gamma(x) &= \underbrace{\epsilon(x)}_{\in P_r,\text{ since }\epsilon \in \mathrm{D}_{P_r}(S)} + \underbrace{b\delta'(x)}_{\in P_r,\text{ since }b \in P_r} \in P_r\end{aligned}$$

Hence, $\gamma(x) \in I' \cap P_r = I$ and this implies that $\gamma \in \mathrm{D}_I(S)$. Finally, we have:

$$\overline{\gamma} = \overline{\delta - a\delta'} = \overline{\delta} \in \mathrm{D}_{I'}(S)/I' \operatorname{Der}_k(S)$$
$$\overline{\gamma} = \overline{\epsilon + b\delta'} = \overline{\epsilon} \in \mathrm{D}_{P_r}(S)/P_r \operatorname{Der}_k(S)$$

Therefore, $\varphi(\overline{\gamma}) = (\overline{\gamma}, \overline{\gamma}) = (\overline{\delta}, \overline{\epsilon})$ and $(\overline{\delta}, \overline{\epsilon}) \in \operatorname{Im}(\varphi)$. ∎

The next step to find an explicit formula for the length of the cokernel of Δ^D, is to apply Proposition 4.4.9 inductively. To keep the formula short, we introduce a new notation:

Definition 4.4.10. Let S be any k-algebra. For ideals $I, J \trianglelefteq S$, denote by $d(I, J)$ the length of $(\mathrm{D}_I(S) + \mathrm{D}_J(S))/(I + J) \operatorname{Der}_k(S)$.

Let S again be a localization of a polynomial ring and I a radical ideal of S. Suppose that I has only one minimal prime, then I is prime and Δ^D is the identity on $\operatorname{Der}_k(S/I)$. Hence, $\operatorname{Coker}(\Delta^D) = 0$. The case, where I has more than one minimal prime ideal, is treated in the following proposition:

Proposition 4.4.11. *Let S be a localization of a polynomial ring over a field k, I a radical ideal of S and $P_1, \ldots, P_r$ the minimal primes of I. Suppose, $r \geq 2$, then we have:*

$$\operatorname{length}_S(\operatorname{Coker}(\Delta^D)) = \sum_{i=2}^{r} d\Big(\bigcap_{j=1}^{i-1} P_j \,,\, P_i\Big)$$

Proof. We do an induction on r, starting with the case $r = 2$. By Proposition 4.4.9, we have an exact sequence:

$$0 \longrightarrow \operatorname{Der}_k(S/I) \xrightarrow{\quad\Delta^D\quad} \operatorname{Der}_k(S/P_1) \times \operatorname{Der}_k(S/P_2) \xrightarrow{\qquad\qquad} \ldots$$

$$\ldots \xrightarrow{\qquad} (\mathrm{D}_{P_1}(S) + \mathrm{D}_{P_2}(S))/(P_1 + P_2) \operatorname{Der}_k(S) \longrightarrow 0$$

Hence, we obtain:

$$\operatorname{Coker}(\Delta^D) \cong (\mathrm{D}_{P_1}(S) + \mathrm{D}_{P_2}(S))/(P_1 + P_2) \operatorname{Der}_k(S)$$

From this, we clearly get that $\operatorname{length}_S(\operatorname{Coker}(\Delta^D)) = d(P_1, P_2)$.

For the case $r \geq 3$, denote by Δ^D_{r-1} the diagonal D-embedding of the ideal $I' = \bigcap_{i=1}^{r-1} P_i$. If we use the identification of the derivation module from Lemma 3.4.11, we get the exact sequence:

$$0 \longrightarrow \mathrm{D}_{I'}(S)/I' \operatorname{Der}_k(S) \xrightarrow{\Delta^D_{r-1}} \prod_{i=1}^{r-1} \mathrm{D}_{P_i}(S)/P_i \operatorname{Der}_k(S) \xrightarrow{\ \beta\ } \operatorname{Coker}(\Delta^D_{r-1}) \longrightarrow 0$$

Replacing Δ^D_{r-1} by the map $\Delta^{D'}_{r-1} = (\Delta^D_{r-1}, id)$ and β by $\beta' = (\beta, 0)$, we obtain another exact sequence:

$$0 \longrightarrow \mathrm{D}_{I'}(S)\big/_{I' \operatorname{Der}_k(S)} \times \mathrm{D}_{P_r}(S)\big/_{P_r \operatorname{Der}_k(S)} \xrightarrow{\Delta^{D'}_{r-1}} \prod_{i=1}^{r} \mathrm{D}_{P_i}(S)\big/_{P_i \operatorname{Der}_k(S)} \xrightarrow{\beta'} \operatorname{Coker}(\Delta^D_{r-1}) \longrightarrow 0$$

No we extend this single sequence to a commutative diagram: the first column map is the embedding $\varphi : \mathrm{D}_I(S)/I \operatorname{Der}_k(S) \hookrightarrow \mathrm{D}_{I'}(S)/I' \operatorname{Der}_k(S) \times \mathrm{D}_{P_r}(S)/P_r \operatorname{Der}_k(S)$ from Proposition 4.4.9. All maps in the first square are induced by the identity. Hence, it is commutative. The second square is also commutative, since $\mathrm{D}_I(S)/I \operatorname{Der}_k(S)$ was embedded into $\operatorname{Ker}(\beta')$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{D}_I(S)\big/_{I \operatorname{Der}_k(S)} \arrow[r] \arrow[d, "\varphi"'] & \mathrm{D}_I(S)\big/_{I \operatorname{Der}_k(S)} \arrow[r] \arrow[d, "\Delta^D"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{D}_{I'}(S)\big/_{I' \operatorname{Der}_k(S)} \times \mathrm{D}_{P_r}(S)\big/_{P_r \operatorname{Der}_k(S)} \arrow[r, "\Delta^{D'}_{r-1}"] & \prod_{i=1}^{r} \mathrm{D}_{P_i}(S)\big/_{P_i \operatorname{Der}_k(S)} \arrow[r, "\beta'"] & \operatorname{Coker}(\Delta^D_{r-1}) \arrow[r] & 0
\end{tikzcd}$$

Since the corresponding kernels of the columns are all 0, the Snake Lemma provides us with the exactness of the Cokernel-sequence:

$$0 \longrightarrow (\mathrm{D}_{I'}(S) + \mathrm{D}_{P_r}(S))/(I' + P_r) \operatorname{Der}_k(S) \longrightarrow \operatorname{Coker}(\Delta^D) \longrightarrow \operatorname{Coker}(\Delta^D_{r-1}) \longrightarrow 0$$

Hence, a recursive formula for the length of $\operatorname{Coker}(\Delta^D)$ can be deduced and we may apply the induction hypothesis:

$$\begin{aligned}
\operatorname{length}_S(\operatorname{Coker}(\Delta^D)) &= \operatorname{length}_S(\operatorname{Coker}(\Delta^D_{r-1})) + d(I', P_r) \\
&= \sum_{i=2}^{r-1} d(\bigcap_{j=1}^{i-1} P_j, P_i) + d(\bigcap_{j=1}^{r-1} P_j, P_r) \\
&= \sum_{i=2}^{r} d(\bigcap_{j=1}^{i-1} P_j, P_i) \qquad \blacksquare
\end{aligned}$$

Now combine Propositions 4.4.8 and 4.4.11. Then we immediately obtain the desired formula for the Deligne number:

Corollary 4.4.12. *Let S be a localized polynomial ring over a field k of characteristic 0. If I is a radical ideal of S so that $\dim S/I = 1$ and $P_1, \ldots, P_r$ are the minimal primes of I, then we have the formula:*

$$m_{S/I} = \sum_{i=1}^{r} m_{S/P_i} + \sum_{i=2}^{r} d(\bigcap_{j=1}^{i-1} P_j\, ,\, P_i\,)$$

In particular:

$$e_{S/I} = 3\delta_{S/I} - \left(\sum_{i=1}^{r} m_{S/P_i} + \sum_{i=2}^{r} d(\bigcap_{j=1}^{i-1} P_j \, , \, P_i \,) \right)$$

5. Computational aspects of invariants

In Chapter 4, we stated algorithmic ideas to compute the conductor and the Deligne number. This Chapter treats algorithms, that we have extracted from these ideas. First of all, we recall some basic algorithms for rings and modules in Section 5.1 and 5.2. Since we need an algorithm for computing the normalization of a ring, we give a short introduction to such algorithms in Section 5.3. The algorithms for the *delta invariant*, the *multiplicity of the conductor* and the *Deligne number* can be found in Section 5.4.
All algorithms stated, work over localizations of polynomial rings. In Section 5.5, we explain, why we can reduce to this case, when we actually want to compute over analytic algebras and give several examples.
At the end of the chapter, we give a short outlook: we describe which future projects could be established by using the implementation and the theory developed in this thesis.

5.1 Basic algorithms for rings and ideals

We will recall some basic algorithms for polynomial rings that were assigned to arbitrary monomial orderings. For the theory of *standard bases* and *normal forms* in this case, we refer to [GP08, Sections 1.6, 1.7].
After a short introduction to the notation used, we will state algorithms for elimination of variables, intersection of ideals, ideal quotients, Krull dimension and vector space dimension of quotient rings.

Definition 5.1.1. Let k be a field and $x = x_1, \ldots, x_n$ indeterminates over k. Let $>$ be a monomial ordering, a total ordering on the set of monomials in x, that satisfies: if $x^\alpha > x^\beta$, then for any γ, we have: $x^{\alpha+\gamma} > x^{\beta+\gamma}$. Denote by LM the leading monomial of a polynomial in $k[x]$ with respect to $>$, then we consider the multiplicatively closed subset of $k[x]$:

$$S_> = \{f \in k[x] \setminus 0 \mid \mathrm{LM}(f) = 1\}$$

Now set:

$$k[x]_> = S_>^{-1} k[x] = \left\{ \frac{g}{f} \,\middle|\, g, f \in k[x], \mathrm{LM}(f) = 1 \right\}$$

So if we pick an arbitrary monomial ordering, the resulting ring is only a localization of a polynomial ring. And we can say even more:

Lemma 5.1.2. *Let k be a field and $>$ a monomial ordering. Then:*

1. $k[x] \subseteq k[x]_> \subseteq k[x]_{\langle x \rangle}$
2. $k[x] = k[x]_>$ *if and only if* $>$ *is global.*
3. $k[x]_{\langle x \rangle} = k[x]_>$ *if and only if* $>$ *is local.*

Proof. See [GP08, Lemma 1.5.2]. ■

To fix notation, let k always denote a field, x, y finite sets of indeterminates over k and R a ring $k[x]_>$, where $>$ is a monomial ordering on $k[x]$.

The first basic algorithm we consider, is the elimination of variables. If we have an ideal I of $R[y]$, then we would like to compute generators of the ideal $I' = I \cap R$. Therefore, we need elimination orderings.

Definition 5.1.3. A monomial ordering on $k[y, x]$ is called an **elimination ordering** for y if it satisfies: for any $f \in k[y, x]$ so that $\mathrm{LM}(f) \in k[x]$, we also have: $f \in k[x]$.

Let us throw a glance at an example before we state the algorithm for computing a standard basis for $I \cap R$.

Example 5.1.4. Let $>$ be an arbitrary monomial ordering on $k[x]$ and $>'$ a global ordering on $k[y]$. Then the Gblock ordering $>_{block} = (>', >)$ is an elimination ordering for y on $k[y, x]$.

Proof. Let $f \in k[y, x]$, so that $\mathrm{LM}(f) \in k[x]$. Then $\mathrm{LM}(f) = x^\alpha$. If $m = y^\beta x^\gamma$ denotes another monomial of f, then we get $\mathrm{LM}(f) >_{block} m$. This means:

$$1 >' y^\beta \text{ or } 1 = y^\beta \text{ and } x^\alpha > x^\gamma$$

Since $>'$ is global, we obtain that $1 = y^\beta$ and therefore, $m \in k[x]$. Hence, $>_{block}$ is an elimination ordering for y. ■

In case of such a block ordering, the example shows that we can choose any global ordering as first component to obtain the elimination property. In the following algorithm, we may therefore choose $(dp, >)$.

In Lemma 5.1.5 we will state, why elimination orderings are the key tool to eliminate variables. In fact, the lemma does not only prove the correctness of Algorithm 1, it also states that a standard basis for the ideal $I \cap R$ is returned.

Algorithm 1 Elimination of variables

Input: An ideal I of $S = R[y] = (k[x]_>)[y]$, where $>$ is an arbitrary monomial ordering on the monomials in x
Output: Generators of the ideal $I' = I \cap R$

1: choose an elimination ordering $>'$ for y on the monomials in y, x that induces $>$ on the monomials in x (i.e. $(dp, >)$)
2: compute a standard basis $\{g_1, \ldots, g_l\}$ of I with respect to $>'$
3: collect those g_i in a set G', whose leading monomials $\mathrm{LM}(g_i)$ do not involve y
4: **return** G'

Lemma 5.1.5. *Let $>$ be an elimination ordering for y on the set of all monomials in y, x and let I be an ideal in $k[y, x]_>$. If G is a standard basis of I, then:*

$$G' = \{g \in G \mid \mathrm{LM}(g) \in k[x]\}$$

is a standard basis of $I' = I \cap k[x]_{>'}$, where $>'$ denotes the monomial ordering on $k[x]$, induced by $>'$.

Proof. See [GP08, Lemma 1.8.3]. ∎

As a consequence of elimination, we are able to compute generators of the intersection of two ideals:

Lemma 5.1.6. *Let $I_1 = \langle f_1, \ldots, f_k \rangle$ and $I_2 = \langle h_1, \ldots, h_r \rangle$ be two ideals of R. Set*

$$J = \langle tf_1, \ldots, tf_k, (1-t)h_1, \ldots, (1-t)h_r \rangle \trianglelefteq R[t]$$

Then: $I_1 \cap I_2 = J \cap R$.

Proof. This is [GP08, Lemma 1.8.10]. ∎

Algorithm 2 Intersection of ideals

Input: $I_1 = \langle f_1, \ldots, f_k \rangle$, $I_2 = \langle h_1, \ldots, h_r \rangle$ ideals of R
Output: Generators of the ideal $I_1 \cap I_2$

1: $J := \langle tf_1, \ldots, tf_k, (1-t)h_1, \ldots, (1-t)h_r \rangle \trianglelefteq R[t]$
2: use Algorithm 1 to compute a set of generators G of $J \cap R$
3: **return** G

The correctness of Algorithm 2 directly follows from Lemma 5.1.6.

Now we may use the intersection of ideals to compute an ideal quotient: $I :_R J$. If $J = \langle h_1, \ldots, h_k \rangle$, then we know from Lemma 4.3.14 that

$$I :_R J = \bigcap_{i=1}^{k} (I :_R \langle h_i \rangle)$$

So we set the focus on quotients: $I :_R \langle h \rangle$. Therefore, consider the following lemma:

Lemma 5.1.7. *Let I be an ideal of R and $h \in R$, $h \neq 0$. Moreover, let $I \cap \langle h \rangle = \langle g_1 h, \ldots, g_k h \rangle$. Then:*

$$I :_R \langle h \rangle = \langle g_1, \ldots, g_k \rangle$$

Proof. See [GP08, Lemma 1.8.12]. ■

The algorithmic idea is clear. Hence, we can state the algorithm for computing the quotient of two ideals:

Algorithm 3 Quotient of two ideals

Input: $I = \langle f_1, \ldots, f_s \rangle$, $J = \langle h_1, \ldots, h_k \rangle$ ideals of R
Output: Generators of the ideal $I :_R J$

1: **for** $i = 1, \ldots, k$ **do**
2: compute a set of generators $G_i = \{g'_{i1}, \ldots, g'_{ik_i}\}$ of $I \cap \langle h_i \rangle$ with 2
3: set $A_i = \langle g_{i1}, \ldots, g_{ik_i} \rangle$, where $g_{ij} = g'_{ij}/h$
4: compute a set of generators G of $\bigcap_{i=1}^{k} A_i$
5: **return** G

For the correctness of Algorithm 3, note that any generating set of $I \cap \langle h_i \rangle$ is of the form $\{g_{i1}h, \ldots, g_{ik_i}h\}$. Hence, we can divide the elements g'_{ij} by h_i and by Lemma 5.1.7, we obtain a generating set for $I :_R \langle h_i \rangle$.

The next step is to compute the Krull dimension and the k-vector space dimension of a quotient R/I. We can reduce this to the case $k[x]/J$, where J is a monomial ideal:

Theorem 5.1.8. *Let I be an ideal of $R = k[x]_>$, then we have:*

a) $\dim(R/I) = \dim(k[x]/L(I))$

b) $\dim_k(R/I) = \dim_k(k[x]/L(I))$ *and if* $\dim_k(R/I) < \infty$, *then the monomials in* $k[x] \setminus L(I)$ *are a k-basis of R/I.*

Proof. See [DL06, Theorem 9.29]. ■

Computing the Krull dimension in this setting reduces to a purely combinatorial task:

Theorem 5.1.9. *If J is a proper ideal of $k[x]$ and G a Gröbner basis of J, then $\dim(k[x]/J)$ is the maximal cardinality of a set $u \subseteq x$ so that no leading monomial of an element of G is in $k[u]$.*

Proof. See [DL06, Theorem 6.3]. ∎

Algorithm 4 Krull dimension of a quotient ring

Input: An ideal I of R
Output: The Krull dimension dimension of R/I

compute a standard basis $G = \{g_1, \ldots, g_k\}$ of I
$u := x$
for $i = 1, \ldots, n$ **do**
    for $j = 1, \ldots, k$ **do**
        if $\mathrm{LM}(g_j) \in k[u]$ **then**
            $u := u \setminus x_i$
            break
$r := \#u$
return r

Theorem 5.1.8 and 5.1.9 prove the correctness of Algorithm 4.

Theorem 5.1.8 also reduces the computation of the k-vector space dimension to a combinatorial question. But in opposition to the Krull dimension of R/I, the vector space dimension can be infinite. So we still need a criterion for finiteness:

Theorem 5.1.10. *Let I be a proper ideal of $k[x]$ and G a Gröbner basis of I. Then the following are equivalent:*

a) $\dim(k[x]/I) = 0$

b) $\dim_k(k[x]/I)$ *is finite.*

c) $\dim_k(k[x]/L(I))$ *is finite.*

d) For each $1 \leq i \leq n$, there exists an $\alpha_i \in \mathbb{N}$ so that $x_i^{\alpha_i}$ is the leading monomial of an element in G.

Proof. See [DL06, Theorem 6.1]. ∎

A way to apply Theorem 5.1.10, is to compute the Krull dimension by Algorithm 4 first and then test it to be zero. After that we count the monomials in $k[x] \setminus L(I)$ since these form a basis of R/I by Theorem 5.1.8.

Algorithm 5 Vector space dimension of quotient ring

Input: An ideal I of R

Output: The k-vector space dimension of R/I

1: compute a standard basis $G = \{g_1, \dots, g_k\}$ of I with respect to $>$
2: compute $d = \dim(R/I)$ by Algorithm 4
3: **if** $d \neq 0$ **then**
4:     **return** ∞
5: pick $\alpha_i \in \mathbb{N}$ so that $x_i^{\alpha_i}$ is the leading monomial of an element of G for $i = 1, \dots, n$
6: $r := 0$
7: Set $L := \{x_1^{j_1} \cdots x_n^{j_n} \mid j_i < \alpha_i\} = \{m_1, \dots, m_l\}$
8: **for** $t = 1, \dots, l$ **do**
9:     **if** $m_t \notin L(I)$ **then**
10:         $r := r + 1$
11: **return** r

Note, that we can efficiently decide, if a monomial m_t is in $L(I)$: we only need to test if m_t is divisible by a leading monomial of the standard basis that we computed before.

Algorithm 5 works correctly and terminates:

Proof. In Step 3, we make use of the finiteness criterion of Theorem 5.1.10. Therefore, the exponents α_i exist and can be chosen. The set L is a finite set of monomials and we know that any monomial in $k[x] \setminus L(I)$ is in L. Hence, we test these monomials to be in $L(I)$ and obtain exactly the set of monomials in $k[x] \setminus L(I)$, which is a basis of R/I by Theorem 5.1.8. ■

5.2 Basic algorithms for modules

In this section, we recall algorithms for modules. First, we state how to compute *syzygies* and generators for the intersection of submodules of a free module. After that, we state an algorithm that returns generators of the kernel of a linear map between modules. Then we are looking for a particular nice way to describe the quotient of two submodules.
Like in the previous section, we denote by k a field, by x a finite set of indeterminates over k and by R a ring $k[x]_>$, where $>$ is an arbitrary monomial ordering on $k[x]$. The vectors $e_1, \dots, e_t$ always denote the canonical basis of R^t.
We refer to [GP08, Section 2.3] for the theory of Gröbner basis and standard basis of free modules R^t.

Definition 5.2.1. Let M be an R-module and $f_1, \ldots, f_k \in M$. A **syzygy** between the f_i is a tuple $(g_1, \ldots, g_k) \in R^k$ so that:

$$\sum_{i=1}^{k} g_i f_i = 0$$

We denote the set of all syzygies between the f_i by $\operatorname{syz}(f_1, \ldots, f_k)$. Note that this is the kernel of the map

$$\begin{aligned} \bigoplus_{i=1}^{k} Re_i &\longrightarrow M \\ e_i &\longmapsto f_i \end{aligned}$$

Therefore, $\operatorname{syz}(f_1, \ldots, f_k)$ is an R-submodule of the free module R^k and it is called **module of syzygies**.

The task is now to state an algorithm for computing generators of the module of syzygies. We will need this, when we deal with intersections of submodules and kernels of linear maps. Another application of syzygies is the computation of free resolutions, see [GP08, Algorithm 2.5.7].
An algorithmic idea can be extracted from the following lemma:

Lemma 5.2.2. *Let $f_1, \ldots, f_k \in R^t$. Consider the canonical embedding*

$$R^t = \bigoplus_{i=1}^{t} Re_i \subseteq \bigoplus_{i=1}^{t+k} Re_i = R^{t+k}$$

and the canonical projection $\pi : R^{t+k} \to R^k$. Let $G = \{g_1, \ldots, g_s\}$ denote a standard basis of $F = \langle f_1 + e_{t+1}, \ldots, f_k + e_{t+k}\rangle$ with respect to an elimination ordering for $e_1, \ldots, e_t$. Assume that $G \cap \bigoplus_{i=t+1}^{t+k} Re_i = \{g_1, \ldots, g_l\}$, then:

$$\operatorname{syz}(f_1, \ldots, f_k) = \langle \pi(g_1), \ldots, \pi(g_l)\rangle$$

Proof. See [GP08, Lemma 2.5.3]. ∎

Remark 5.2.3. The lemma requires an *elimination ordering* for $e_1, \ldots, e_t$. Let $>$ be an arbitrary monomial ordering on $k[x]$, then an example for an elimination ordering for $e_1, \ldots, e_t$ is given by $>_{el}$:

$$x^\alpha e_i >_{el} x^\beta e_j \text{ if } i < j \text{ or } i = j \text{ and } x^\alpha > x^\beta$$

The name of the ordering comes from a fact that is similar to elimination orderings for variables: let $f \in R$ so that $\operatorname{LM}(f) \in \bigoplus_{i=t+1}^{t+k} k[x]e_i$, then $f \in \bigoplus_{i=t+1}^{t+k} Re_i$.

Now that we know, how to construct an elimination ordering, we are able to state the algorithm.

Algorithm 6 Generators for module of syzygies

Input: $f_1, \dots, f_k \in R^t$

Output: Generators of $\mathrm{syz}(f_1, \dots, f_k)$

1: $F := \{f_1 + e_{t+1}, \dots, f_k + e_{t+k}\}$, where $R^{t+k} = \bigoplus_{i=1}^{t+k} Re_i \supseteq \bigoplus_{i=1}^{t} Re_i = R^t$

2: compute a standard basis $G = \{g_1, \dots, g_s\}$ of $\langle F \rangle$ with respect to an elimination ordering for $e_1, \dots, e_t$

3: let $\{g_1, \dots, g_l\} = G \cap \bigoplus_{i=t+1}^{t+k} Re_i$, where $g_i = \sum_{j=1}^{k} g_{ij} e_{t+j}$, $i = 1, \dots, l$

4: set $g_i' := (g_{i1}, \dots, g_{ik})$, $i = 1, \dots, l$

5: **return** $\{g_1', \dots, g_l'\}$

This can also be found in [GP08, Algorithm 2.5.4]. The correctness follows from from Lemma 5.2.2
The algorithm can even be extended to the case, where we compute over a quotient ring $S = R/I$. This is due to [GP08, Remark 2.5.6].

The first application of Algorithm 6 is the computation of generators of an intersection of submodules. We make use of the following lemma:

Lemma 5.2.4. *Let $f_1, \dots, f_k$ and $g_1, \dots, g_s \in k[x]^t$. Set $U = \langle f_1, \dots, f_k \rangle R^t$ and $V = \langle g_1, \dots, g_s \rangle R^t$. Denote by $c_1, \dots, c_{t+k+s} \in k[x]^{2t}$ be the columns of the matrix:*

$$\left[\begin{array}{ccc|c|c} 1 & & & & \\ & \ddots & & f_1 \dots f_k & 0 \dots 0 \\ & & 1 & & \\ \hline 1 & & & & \\ & \ddots & & 0 \dots 0 & g_1 \dots g_s \\ & & 1 & & \end{array}\right]$$

Then for $h \in k[x]^t$, we have: $h \in U \cap V$ if and only if h appears as the first t components of an element $h' \in \mathrm{syz}(c_1, \dots, c_{t+k+s})$.

Proof. Let h be in $U \cap V \subseteq R^t$. Then we can write:

$$h = \sum_{i=1}^{t} h_i e_i = \sum_{l=1}^{k} r_l f_l = \sum_{j=1}^{s} r_j' g_j,$$

where $h_i \in k[x]$ and $r_l, r_j' \in R$. The tuple

$$(h_1, \dots, h_t, -r_1, \dots, -r_k, -r_1', \dots, -r_s')$$

is in $\mathrm{syz}(c_1, \dots, c_{t+k+s})$ and h appears as the first t components of it.
If $h = \sum_{i=1}^{t} h_i e_i \in k[x]^t$ so that there is an element

$$(h_1, \dots, h_t, r_1, \dots, r_k, r_1', \dots, r_s') \in \mathrm{syz}(c_1, \dots, c_{t+k+s}),$$

then we immediately get equations:

$$0 = \sum_{i=1}^{t} h_i e_i + \sum_{l=1}^{k} r_l f_l \text{ and } 0 = \sum_{i=1}^{t} h_i e_i + \sum_{j=1}^{s} r'_j g_j$$

Hence, $h \in U \cap V$. ■

This can be turned into an algorithm and the correctness can be deduced from Lemma 5.2.4. But note that the submodules have to be generated by elements of $k[x]^t$.

Algorithm 7 Intersection of submodules

Input: $U = \langle f_1, \ldots, f_k \rangle R^t$ and $V = \langle g_1, \ldots, g_s \rangle R^t$, where $f_i, g_j \in k[x]^t$
Output: A set of generators of $U \cap V$

1: Let $c_1, \ldots, c_{t+k+s}$ as in Lemma 5.2.4
2: compute a set of generators $H = \{h_1, \ldots, h_m\}$ of $\mathrm{syz}(c_1, \ldots, c_{t+k+s})$ with Algorithm 6
3: Set $h'_i = \pi_t(h_i)$, $i = 1, \ldots, m$, where π_t denotes the projection to the first t components
4: **return** $H' = \{h'_1, \ldots, h'_m\}$

Our second application of the module of syzygies are kernels of linear maps. Since we allow these computations over quotient rings, we may fix notations: $S = R/I$, where I is an ideal of R. We first state an algorithm and discuss afterwards why it is correct.

Algorithm 8 Generators for kernel of a linear map

Input: A matrix $B = (b_1, \ldots, b_k)$, representing a linear map $\varphi : S^k/U \to S^m/V$, where U is a submodule of S^k and $V = \langle v_1, \ldots, v_s \rangle$ is a submodule of S^m
Output: Generators for $\mathrm{Ker}(\varphi)$ in S^k

1: compute generators $h_1, \ldots, h_l$ for the syzygy-module:

$$syz(b_1, \ldots, b_k, v_1, \ldots, v_s) \subseteq S^{k+s}$$

2: set $h'_i = \pi_k(h_i)$, $i = 1, \ldots, l$, where π_k denotes the projection to the first k components
3: **return** $\{h'_1, \ldots, h'_l\}$

The reason why Algorithm 8 works correctly is due to the following remark, which can also be found in [GP08, 2.8.7].

Remark 5.2.5. Let U be a submodule of S^k and $V = \langle v_1, \dots, v_s \rangle$ a submodule of S^m. Let $\varphi : S^k/U \to S^m/V$ be an S-linear map, represented by the matrix $B = (b_1, \dots, b_k)$, where $b_i \in S^m$.
We want to compute generators in S^k for $\mathrm{Ker}(\varphi)$ by making use of syzygies. An element $f = \sum_{i=1}^k f_i e_i$ is in $\mathrm{Ker}(\varphi)$ if and only if there exist $y_1, \dots, y_s \in S$ so that $\sum_{i=1}^k f_i b_i = \sum_{j=1}^s y_j v_j$ and this means that $(f_1, \dots, f_k, -y_1, \dots, -y_s) \in \mathrm{syz}(b_1, \dots, b_k, v_1, \dots, v_s)$.

Now that we are able to compute kernels of such maps, we may state an algorithm that returns a useful representation of a quotient of two submodules. In Singular, this algorithm is available via the command `modulo`.

Algorithm 9 Representation of submodule-quotient

Input: Two submodules $U = \langle u_1, \dots, u_k \rangle, V = \langle v_1, \dots, v_s \rangle$ of the free module S^m
Output: Generators of a module C, so that $S^k/C \cong (U+V)/V$

1: define the matrix $B = (u_1, \dots, u_k)$, representing the map $\varphi_U : S^k \to S^m/V$
2: compute generators $c_1, \dots, c_t$ of $C = \mathrm{Ker}(\varphi_U)$ with Algorithm 8
3: **return** $\{c_1, \dots, c_t\}$

The algorithm computes the desired representation:

Proof. Algorithm 8 returns the generators $c_1, \dots, c_t$ of $C = \mathrm{Ker}(\varphi_U)$ and by the homomorphism theorem we get:

$$S^k/C \cong \mathrm{Im}(\varphi_U) = (U+V)/V$$

■

Our last consideration in this section is an extension of an algorithm that we examined in Section 5.1: the Singular command `vdim`.

Remark 5.2.6. In Algorithm 5, we have seen that we are able to compute the vector space dimension of quotient rings. It is possible to extend this algorithm so that we are able to compute the vector space dimension of quotients R^t/M, where M is a submodule of R^t. In Singular, this is implemented via the command `vdim`. We refer to the manual of [Singular] for details.

5.3 Computing the normalization

We want to compute the normalization of a reduced ring S. Therefore, we will give an introduction to the algorithmic idea and then chose an algorithm, that is suitable for our considerations. A first step is to define the notion of *test ideals*. These are used to construct chains of rings between S and $\overline{S}$. Such chains will stabilize, which ensures the termination of the algorithm and with the *Grauert-Remmert criterion*, we will also get the correctness.

Definition 5.3.1. An ideal J of S is called **test ideal** of S if it satisfies:

- J contains a non-zero divisor of S,
- $\sqrt{J} = J$,
- $N(S) \subseteq V(J)$, where $N(S)$ denotes the non-normal locus of S:

$$N(S) = \{P \in \mathrm{Spec}(S) \mid S_P \text{ is not normal}\}$$

We mentioned at the beginning of this section, that test ideals generate rings between S and $\overline{S}$:

Lemma 5.3.2. *Let J be a test ideal of S. Then we have inclusions:*

$$S \subseteq \mathrm{Hom}_S(J, J) \subseteq \overline{S}$$

Proof. See [GP08, Lemma 3.6.1]. ■

The criterion which allows us to state the algorithmic idea is due to Grauert and Remmert. It relates the endomorphism rings of the test ideals to the normalization.

Proposition 5.3.3. *Let J be a test ideal of S. Then, S is normal if and only if $S = \mathrm{Hom}_S(J, J)$.*

Proof. This is [GP08, Proposition 3.6.5]. ■

Remark 5.3.4. The algorithmic idea to compute $\overline{S}$ combines Lemma 5.3.2 and the criterion given in Proposition 5.3.3.
Let J_0 be a test ideal of $S_0 = S$. Then we compute $\mathrm{Hom}_{S_0}(J_0, J_0) = S_1$ and we know from Lemma 5.3.2 that: $S_0 \subseteq S_1 \subseteq \overline{S}$.

- If $S_0 = S_1$, then we may apply Proposition 5.3.3 to get that S_0 is normal.
- If $S_0 \subsetneq S_1$, then S_0 cannot be normal. Therefore, we may succeed with S_1 in the same way. Note, that $Q(S_1) = Q(S_0)$ and thus, $\overline{S_1} = \overline{S_0}$ by Proposition 2.1.13.

We get a chain of rings in $\overline{S}$:

$$S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i \subseteq \cdots \subseteq \overline{S}$$

This has to stabilize since $\overline{S}$ is a finitely generated S-module. Therefore, there exists an index $l \in \mathbb{N}$ so that: $S_l = S_{l+1}$. But $S_{l+1} = \mathrm{Hom}_{S_l}(J_l, J_l)$ for a test ideal J_l in S_l. Hence, S_l is normal by Proposition 5.3.3 and we get that $\overline{S} = \overline{S_l} = S_l$.

An algorithm that follows this idea can be found in [GP08, Algorithm 3.6.9]. It computes the normalization in the case, where $S = k[x]/I$ is an integral domain. Since we know by Proposition 2.1.15 that we can reduce to this case, we just need to find the minimal associated primes. For this task, we refer to [GP08, Chapter 4].

Another normalization algorithm can be found in [GLS10]. Let $R = k[x]$, where k is a perfect field and I a prime ideal in R. Algorithm [GLS10, Algorithm 3] computes an ideal U of R and $d \in R$ so that $\overline{R/I} = \frac{1}{d}U$ in $Q(R/I)$. So we get the R/I-module generators of $\overline{R/I}$.
In [GLS10, Lemma 6.1] and [GLS10, Remark 6.3], the authors generalize the algorithm to non-global orderings. This basically works since we can swap localization and normalization by Proposition 2.1.17.

Remark 5.3.5. For an implementation of [GLS10, Algorithm 3], we refer to the [Singular] manual and the command `normal`. This also handles the case, where I is not necessarily a prime ideal but only a radical ideal. The algorithm computes the minimal primes $P_1, \dots, P_r$, ideals $U_1, \dots, U_r \trianglelefteq R$, $U_i = \langle u_1^{(i)}, \dots, u_{k_i}^{(i)} \rangle$ and $d_1, \dots, d_r \in R$ so that $\overline{R/P_i} = \frac{1}{d_i}U_i$ in $Q(R/P_i)$, for $i = 1, \dots, r$. It also returns representations of $\overline{R/P_i}$ as R-algebras: rings $R[t_i] = [t_1^{(i)}, \dots, t_{k_i}^{(i)}]$ and ideals $J_i \trianglelefteq R[t_i]$ so that $R[t_i]/J_i = \overline{R/P}$. Note, that the $t_j^{(i)}$ correspond to the elements $\frac{u_j^{(i)}}{d_i}$ since J_i is the kernel of the map $R[t_i] \to \frac{1}{d_i}U_i$.

5.4 Algorithms for computing invariants

The first invariant that we compute, is the delta invariant of R/I, where $R = k[x]_>$, $>$ a local ordering and I a radical ideal of R. We reduce the computation of $\delta_{R/I}$ to that of a single branch: R/P, where P is a minimal associated prime ideal of I. After that, we state algorithms for the conductor and the multiplicity of the conductor. The algorithmic idea to compute $\mathfrak{C}_{R/I}$ and $c_{R/I}$ is taken from Section 4.3. At last, we also turn the ideas from Section 4.4 into an algorithm to compute the Deligne number $e_{R/I}$.
Implementations of the stated algorithms in Singular, except for the delta invariant, which is already implemented as part of the command `normal`, can be found in Appendix B.

Lemma 5.4.1. *Let I be a radical ideal of R and $I = \bigcap_{i=1}^r P_i$ the decomposition into its minimal primes. Denote by (U_i, d_i) the output of the normalization algorithm applied to R/P_i. Then we have:*

a) $\delta_{R/P_i} = \dim_k(U_i/d_iU_i)$ *for all* $i = 1, \dots, r$.

b) $\delta_{R/I} = \sum_{i=1}^r \delta_{R/P_i} + \sum_{i=1}^{r-1} \dim_k(R/(I + \bigcap_{j=i+1}^r P_j))$

Proof. This is a special case of [GLS10, Lemma 4.7]. ■

In fact, Lemma 5.4.1 reduces the computation of $\delta_{R/I}$ to the branch-case since we can compute the vector space dimension of $R/(I + \bigcap_{j=i+1}^{r} P_j)$ by Algorithm 5. Applying Algorithm 9 to U_i/d_iU_i, we obtain a representation: $R^{t_i}/C_i \cong U_i/d_iU_i$. Hence, we can also compute the vector space dimension of U_i/d_iU_i by Remark 5.2.6. Altogether, we can state the following algorithm whose correctness directly follows from Lemma 5.4.1.

Algorithm 10 Delta invariant

Input: A radical ideal I of R so that $\dim R/I = 1$ and its minimal primes $P_1, \ldots, P_r$

Output: The delta invariant $\delta_{R/I}$

apply the normalization algorithm to P_i. Denote the output by (U_i, d_i)
for $i = 1, \ldots, r$ **do**
    compute C_i and l_i, so that $U_i/d_iU_i \cong R^{l_i}/C_i$ using Algorithm 9
    compute $\delta_i := \dim_k(R^{l_i}/C_i)$
    compute $I^{(i)} := \bigcap_{j=i+1}^{r} P_j$ using Algorithm 2
    compute $\gamma_i := \dim_k(R/(I + I^{(i)}))$
return $\sum_{i=1}^{r} \delta_i + \sum_{i=1}^{r-1} \gamma_i$

Now focus on the computation of the conductor $\mathfrak{C}_{R/I}$. Theorem 4.3.16 states a computable form of $\mathfrak{C}_{R/I}$ and therefore allows us to derive an algorithm. In fact, this algorithm is independent of the chosen ordering on $R = k[x]_>$.

Algorithm 11 Conductor

Input: A radical ideal I of R

Output: The conductor $\mathfrak{C}_{R/I}$ as an ideal in R

apply the normalization algorithm to I. Denote the output by (U_i, d_i) and let $R[t_i]/J_i$ be the R-algebra representation of $\overline{R/P_i}$, where P_i is a minimal prime of I, $i = 1, \ldots, r$
compute $P_i = J_i \cap R$ by Algorithm 1, $i = 1, \ldots, r$
use Algorithms 2 and 3 to compute the ideal

$$I^{(i)} := \left(P_i + d_i \bigcap_{j \neq i} P_j \right) :_R U_i$$

for $i = 1, \ldots, r$
return $\bigcap_{i=1}^{r} I^{(i)}$

Note, that the ideals $P_i = J_i \cap R$ are indeed the minimal primes ideals of I. This can be seen in the first part of the proof of Theorem 4.3.16. Hence, the correctness follows by this theorem.

The next algorithm computes the multiplicity of the conductor $c_{R/I}$. Therefore, we make use of the formula stated in Remark 4.3.11:

Algorithm 12 Multiplicity of the conductor

Input: A radical ideal I of R so that $\dim R/I = 1$
Output: The multiplicity of the conductor: $c_{R/I}$

1: use Algorithm 10 to compute $\delta_{R/I}$
2: compute the conductor $\mathfrak{C}_{R/I} \trianglelefteq R$ with Algorithm 11
3: compute $d := \dim_k(R/\mathfrak{C}_S)$
4: **return** $\delta_{R/I} + d$

For the correctness, note that the conductor $\mathfrak{C}_{R/I}$ is an ideal in R. In fact, it is the preimage of the conductor ideal of $S = R/I$. Hence, $R/\mathfrak{C}_S \cong S/\overline{\mathfrak{C}_S}$ and we get: $d = \dim_k(S/\overline{\mathfrak{C}_S})$. Then the formula of Remark 4.3.11 shows that the algorithm works correctly.

The Deligne number $e_{R/I}$ depends on the colength of derivations and the delta invariant. Since we can compute the latter by Algorithm 10, we focus on the calculation of the colength of derivations. Like for $\delta_{R/I}$, we reduce to the branch-case. Therefore, we make use of the formula that we derived in Corollary 4.4.12:

$$m_{R/I} = \sum_{i=1}^{r} m_{R/P_i} + \sum_{i=2}^{r} d\Big(\bigcap_{j=1}^{i-1} P_j\,,\, P_i\,\Big),$$

where $P_1, \ldots, P_r$ are the minimal primes of I.

The first step is to compute $d(I, J)$, where I, J are ideals in R. But therefore, we need generators of the I and J-preserving derivations. The following lemma describes how we can find them:

Lemma 5.4.2. *Let $I = \langle f_1, \ldots, f_k \rangle$ be an ideal of $R = k[x]_> = k[x_1, \ldots, x_n]_>$. Set*

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & f_1 & & 0 & \cdots & f_k & & 0 \\ \vdots & & \vdots & & \ddots & & & & \ddots & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} & 0 & & f_1 & \cdots & 0 & & f_k \end{bmatrix}$$

and denote by $\Phi : R^{n+kn} \to R^k$ the linear map induced by A. Consider

$R^n \subseteq R^{n+kn}$ as the submodule generated by the first n components, then:

$$\mathrm{D}_I(R) \cong \mathrm{Ker}(\Phi) \cap R^n$$

Proof. The proof of [Epu15, Lemma 6.25] treats the case, where $R = k[x]$. Taking a closer look at the proof, we see that the arguments also work for $R = k[x]_>$. ■

The resulting algorithm is also due to [Epu15] and the correctness follows from Lemma 5.4.2.

Algorithm 13 I-preserving derivations

Input: An ideal $I = \langle f_1, \ldots, f_k \rangle$ of R
Output: A generating set G of $\mathrm{D}_I(R)$

1: denote by $c_1, \ldots, c_{n+kn}$ the columns of the matrix of Lemma 5.4.2
2: compute generators $\{g'_1, \ldots, g'_l\}$ of $\mathrm{syz}(c_1, \ldots, c_{n+kn})$ using Algorithm 6
3: set $g_i = \pi_n(g'_i)$, where π_n denotes the projection to the first n components
4: **return** $G := \{g_1, \ldots, g_l\}$

As a consequence, we are now able to compute $d(I, J)$:

Algorithm 14 $d(I, J)$

Input: Ideals $I = \langle f_1, \ldots, f_k \rangle, J = \langle g_1, \ldots, g_l \rangle$ of R
Output: $d(I, J)$

1: compute sets of generators G_I and G_J of $\mathrm{D}_I(R)$ and $\mathrm{D}_J(R)$ using Algorithm 13
2: set $N := (I + J)R^n$ and $M := \mathrm{D}_I(R) + \mathrm{D}_J(R)$
3: compute generators of a module C and $t \in \mathbb{N}$ so that $R^t/C \cong M/N$
4: compute $d := \dim_k(R^t/C)$
5: **return** d

Algorithm 14 works correctly:

Proof. The algorithm follows the definition of the d-notation.

$$\begin{aligned} d(I, J) &= \mathrm{length}_R((\mathrm{D}_I(R) + \mathrm{D}_J(R))/(I + J)\,\mathrm{Der}_k(R)) \\ &= \mathrm{length}_R((\mathrm{D}_I(R) + \mathrm{D}_J(R))/(I + J)R^n) \\ &= \mathrm{length}_R(M/N) \\ &= \mathrm{length}_R(R^t/C) \\ &= \dim_k(R^t/C) \\ &= d \end{aligned}$$

Note that we used: $\mathrm{Der}_k(R) \cong R^n$. This is due to Example 3.4.3. ■

The second step to compute the Deligne number is to state an algorithm for a single colength of derivations $m_{R/P}$, where P is a prime ideal. We have to simulate the injection $\operatorname{Der}_k(R/P) \hookrightarrow \operatorname{Der}_k(\overline{R/P})$ to compute the length of $\operatorname{Der}_k(\overline{R/P}) / \operatorname{Der}_k(R/P)$. The solution to this will be the quotient rule. For the algorithm, we also need the following: If $f \in I = \langle f_1, \dots, f_l \rangle \trianglelefteq R$, then we can compute polynomials $u, g_1, \dots, g_l \in k[x]$ so that: $u \in R^*$ and

$$uf = \sum_{i=1}^{l} g_i f_i$$

For this we refer to [GP08, Section 2.8.1] and the Singular command `division`.

Algorithm 15 Colength of derivations of a prime ideal

Input: A prime ideal P of R so that $\dim R/P = 1$
Output: $m_{R/P}$

apply the normalization algorithm to P. Denote the output by (U, d). Let $U = \langle u_1, \dots, u_k \rangle$ and $R[t]/J = R[t_1, \dots, t_k]/J$ be the R-algebra presentation of $\overline{R/P}$
compute a set of generators $\delta_1, \dots, \delta_r$ of $\mathrm{D}_P(R)$ using Algorithm 13
for $i = 1, \dots, r$ **do**
  for $j = 1, \dots, k$ **do**
    set $f = \delta_i(u_j)d - \delta_i(d)u_j$
    use `division` to get $d_{ij}, g_{ij} \in k[x,t]$, $a \in J$ so that:

$$d_{ij} \in R[t]^* \text{ and}$$
$$d_{ij} f = g_{ij} d^2 + a$$

compute $D = \prod_{i,j} d_{ij} \in k[x,t] \cap R[t]^*$
for $i = 1, \dots, r$ **do**
  set $\delta_i^{ext} := D\delta_i$
  for $j = 1, \dots, k$ **do**
    $\delta_i^{ext} := \delta_i^{ext} + \frac{D}{d_{ij}} g_{ij} \frac{\partial}{\partial_{t_j}}$
set $M^{ext} = \langle \delta_1^{ext}, \dots, \delta_r^{ext} \rangle$
compute generators $\gamma_1, \dots, \gamma_z$ of $\mathrm{D}_J(R[t])$ using Algorithm 13
compute module C so that $R[t]^w/C \cong \mathrm{D}_J(R[t])/(JR[t]^{n+k} + M^{ext})$ with Algorithm 9
compute $m = \dim_k(R[t]^w/C)$
return m

Algorithm 15 works correctly:

Proof. First, we set $\varphi : \mathrm{Der}_k(R/P) \to \mathrm{Der}_k(\overline{R/P})$. This is the embedding of Proposition 4.4.4. In Step 2, we compute the generators of $\mathrm{Der}_k(R/P) = \mathrm{D}_P(R)/P\,\mathrm{Der}_k(R)$, these are $\overline{\delta_1}, \ldots, \overline{\delta_r}$. Now apply φ to $\overline{\delta_i}$, then we get a derivation $\varphi(\overline{\delta_i}) : \overline{R/P} \to \overline{R/P}$. Since $\overline{R/P} = R[t]/J$, we can also consider $\varphi(\overline{\delta_i})$ as a class $\overline{\epsilon_i}$ in $\mathrm{D}_J(R[t])/J\,\mathrm{Der}_k(R[t])$, where $\epsilon_i \in \mathrm{D}_J(R[t])$. Hence, we obtain:

$$\mathrm{Der}_k(R/P) \cong \mathrm{Im}(\varphi) = \langle \overline{\epsilon_1}, \ldots, \overline{\epsilon_r} \rangle$$

The next step is to show, that the algorithm computes representatives of the $\overline{\epsilon_i}$, up to the unit D. For this, note that $\overline{\epsilon_i}$ is the unique extension of $\overline{\delta_i}$ to $R[t]/J$. Using that $\overline{t_j} = \frac{\overline{u_j}}{\overline{d}} \in R[t]/J$ by Remark 5.3.5, we can deduce:

$$\overline{\epsilon_i}(\overline{t_j}) = \overline{\epsilon_i}\left(\frac{\overline{u_j}}{\overline{d}}\right) = \frac{\overline{\delta_i}(\overline{u_j})\overline{d} - \overline{\delta_i}(\overline{d})\overline{u_j}}{\overline{d^2}} = \frac{\overline{\delta_i(u_j)d - \delta_i(d)u_j}}{\overline{d^2}} \in R[t]/J$$

Thus, $\delta_i(u_j)d - \delta_i(d)u_j \in \langle d^2 \rangle + J \trianglelefteq R[t]$. So we can actually perform Step 6 of the algorithm: we get $d_{ij}, g_{ij} \in k[x,t]$ and $a \in J$ so that $d_{ij} \in R[t]^*$ and $d_{ij}(\delta_i(u_j)d - \delta_i(d)u_j) = g_{ij}d^2 + a$. Hence, we get the equality:

$$\frac{\overline{g_{ij}}}{\overline{d_{ij}}} = \frac{\overline{\delta_i(u_j)d - \delta_i(d)u_j}}{\overline{d^2}} = \overline{\epsilon_i}(\overline{t_j})$$

In the steps 8 to 11, the algorithm constructs

$$\delta_i^{ext} = D\delta_i + \sum_{j=1}^{k} \frac{D}{d_{ij}} g_{ij} \frac{\partial}{\partial_{t_j}}$$

Then $\delta_i^{ext} \in \mathrm{Der}_k(R[t])$, since $R[t] = k[x]_>[t] = k[x,t]_{(>,>')}$, where $>'$ is global. Therefore, $R[t]$ is just a localization of $k[x,t]$ and by Example 3.4.3, we get: $\mathrm{Der}_k(R[t]) = \bigoplus_{l=1}^{n} R[t]\frac{\partial}{\partial_{x_l}} \oplus \bigoplus_{j=1}^{k} R[t]\frac{\partial}{\partial_{t_j}}$.
Now consider in $R[t]/J$:

$$\begin{aligned} \overline{\delta_i^{ext}(x_l)} &= \overline{D\delta_i(x_l)} = \overline{D} \cdot \overline{\delta_i}(\overline{x_l}) = \overline{D} \cdot \overline{\epsilon_i}(\overline{x_l}) = \overline{D\epsilon_i(x_l)} \\ \overline{\delta_i^{ext}(t_j)} &= \frac{\overline{D}}{\overline{d_{ij}}}\overline{g_{ij}} = \overline{D} \cdot \overline{\epsilon_i}(\overline{t_j}) = \overline{D\epsilon_i(t_j)} \end{aligned}$$

Hence, there exist $c_{il}, c'_{ij} \in J$ so that $\delta_i^{ext}(x_l) = D\epsilon_i(x_l) + c_{il}$ and $\delta_i^{ext}(t_j) = D\epsilon_i(t_j) + c'_{ij}$. As a consequence, we can write:

$$\begin{aligned} \delta_i^{ext} &= \sum_{l=1}^{n}(D\epsilon_i(x_l) + c_{il})\frac{\partial}{\partial_{x_l}} + \sum_{j=1}^{k}(D\epsilon_i(t_j) + c'_{ij})\frac{\partial}{\partial_{t_j}} \\ &= \underbrace{D\epsilon}_{\in \mathrm{D}_J(R[t])} + \underbrace{\sum_{l=1}^{n} c_{il}\frac{\partial}{\partial_{x_l}} + \sum_{j=1}^{k} c'_{ij}\frac{\partial}{\partial_{t_j}}}_{\in J\,\mathrm{Der}_k(R[t]) \;\subseteq\; \mathrm{D}_J(R[t])} \end{aligned}$$

We obtain that $\delta_i^{ext} \in \mathrm{D}_J(R[t])$ and in $\mathrm{D}_J(R[t])/J\operatorname{Der}_k(R[t])$, we have the equality: $\overline{\delta_i^{ext}} = \overline{D\epsilon_i}$.
The last thing to show is that the dimension m, computed in Step 15, is really equal to $m_{R/P}$. We set $M^{ext} = \langle \delta_1^{ext}, \ldots, \delta_r^{ext} \rangle$ in Step 12. With the equality $\overline{\delta_i^{ext}} = \overline{D\epsilon_i}$, we can deduce that

$$\begin{aligned}(M^{ext} + J\operatorname{Der}_k(R[t]))/J\operatorname{Der}_k(R[t]) &= \overline{M^{ext}} \\ &= \overline{D}\langle \overline{\epsilon_1}, \ldots, \overline{\epsilon_r} \rangle \\ &\cong \langle \overline{\epsilon_1}, \ldots, \overline{\epsilon_r} \rangle \\ &\cong \operatorname{Der}_k(R/P)\end{aligned}$$

With the identification $\operatorname{Der}_k(R[t]) \cong R[t]^{n+k}$ and Step 14, we finally obtain:

$$\begin{aligned}&\operatorname{Der}_k(\overline{R/P})/\operatorname{Der}_k(R/P) = \operatorname{Der}_k(R[t]/J)/\operatorname{Der}_k(R/P) \\ &\cong \mathrm{D}_J(R[t])/J\operatorname{Der}_k(R[t])\Big/(M^{ext} + J\operatorname{Der}_k(R[t]))/J\operatorname{Der}_k(R[t]) \\ &\cong \mathrm{D}_J(R[t])/(M^{ext} + J\operatorname{Der}_k(R[t])) \\ &\cong \mathrm{D}_J(R[t])/(M^{ext} + JR[t]^{n+k}) \\ &\cong R[t]^w/C\end{aligned}$$

Therefore, $m = \dim_k(R[t]^w/C) = m_{R/P}$. ■

To finally compute the Deligne number, we combine the algorithms 14 and 15:

Algorithm 16 Deligne number

Input: A radical ideal I of R so that $\dim R/I = 1$
Output: $e_{R/P}$

```
compute the minimal primes P_1, ..., P_r of I
compute δ_{R/I} using Algorithm 10
for i = 1, ..., r do
    compute m_{R/P_i} using Algorithm 15
    compute I^{(i)} = ⋂_{j=1}^{i-1} P_j using Algorithm 2
    compute d_i = d(I^{(i)}, P_i) using Algorithm 14
return 3δ_{R/I} − (∑_{i=1}^{r} m_{R/P_i} + ∑_{i=2}^{r} d_i)
```

The correctness follows from Corollary 4.4.12.

5.5 Justification and examples

The original idea of this thesis was to compute invariants of curve singularities $\mathbb{C}\{x\}/I$ but the algorithms stated in Section 5.4 only work over localized

polynomial rings. In this section, we will make use of the stability results that we have established in Chapter 4 to reduce the computation of invariants of $\mathbb{C}\{x\}$ to the case $\mathbb{Q}[x]_{\langle x\rangle}$, which we can handle with the stated algorithms. At the end of this section, we throw a glance at some examples.
In the following, let I always denote a radical ideal of $\mathbb{Q}[x]_{\langle x\rangle}$ so that the ring $\mathbb{Q}[x]_{\langle x\rangle}/I$ is of dimension 1.

Theorem 5.5.1. *The delta invariant of* $\mathbb{Q}[x]_{\langle x\rangle}/I$ *and* $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$ *coincide:*

$$\delta_{\mathbb{Q}[x]_{\langle x\rangle}/I} = \delta_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}}$$

Proof. The ring $S = \mathbb{Q}[x]_{\langle x\rangle}/I$ is a reduced excellent local algebra over the perfect field $\mathbb{Q}$ with residue field $\mathbb{Q}$. Hence, we apply Proposition 4.2.4 and obtain: $\delta_S = \delta_T$, where $T = (S \otimes_{\mathbb{Q}} \mathbb{C})_{\langle x\rangle} = \mathbb{C}[x]_{\langle x\rangle}/I\mathbb{C}[x]_{\langle x\rangle}$. By Lemma 1.3.8, the resulting ring T is still reduced since $\mathbb{Q}$ is perfect and S is reduced. Now we apply Proposition 4.2.3 and obtain for the $\langle x\rangle$-adic completion:

$$\delta_T = \delta_{\widehat{T}}$$

We have that $\widehat{T} = \mathbb{C}[[x]]/I\mathbb{C}[[x]]$ is still reduced by Theorem 2.4.14. If we apply Proposition 4.2.3 to $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$, we get:

$$\delta_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}} = \delta_{\mathbb{C}[[x]]/I\mathbb{C}[[x]]}$$

Note, that $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$ is reduced, since we can consider it as a subring of its reduced completion $\mathbb{C}[[x]]/I\mathbb{C}[[x]]$ by Corollary 1.4.17.
Altogether, we obtain the desired equality:

$$\delta_{\mathbb{Q}[x]_{\langle x\rangle}/I} = \delta_S = \delta_T = \delta_{\widehat{T}} = \delta_{\mathbb{C}[[x]]/I\mathbb{C}[[x]]} = \delta_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}}$$

■

The second invariant that we have considered, is the multiplicity of the conductor. Note that the conductor behaves well with respect to field extension, localization and completion but is not stable.

Theorem 5.5.2. *The multiplicity of the conductor of* $\mathbb{Q}[x]_{\langle x\rangle}/I$ *and* $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$ *coincide:*

$$c_{\mathbb{Q}[x]_{\langle x\rangle}/I} = c_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}}$$

Proof. We argue as before in Theorem 5.5.1: denote by $S = \mathbb{Q}[x]_{\langle x\rangle}/I$, by $T = \mathbb{C}[x]_{\langle x\rangle}/I\mathbb{C}[x]_{\langle x\rangle}$ and by $W = \mathbb{C}[[x]]/I\mathbb{C}[[x]]$. Applying Proposition 4.3.13, we obtain $c_S = c_T$ and by Proposition 4.3.12, we get: $c_T = c_W$. If we apply Proposition 4.3.12 again to $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$, we also have: $c_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}} = c_W$. Therefore, we can deduce that

$$c_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}} = c_S$$

■

Now we state a similar theorem for the Deligne number:

Theorem 5.5.3. *The Deligne number of* $\mathbb{Q}[x]_{\langle x\rangle}/I$ *and* $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$ *coincide:*

$$e_{\mathbb{Q}[x]_{\langle x\rangle}/I} = e_{\mathbb{C}\{x\}/I\mathbb{C}\{x\}}$$

Proof. Denote by $S = \mathbb{Q}[x]_{\langle x\rangle}/I$, by $T = \mathbb{C}[x]_{\langle x\rangle}/I\mathbb{C}[x]_{\langle x\rangle}$ and by $W = \mathbb{C}[[x]]/I\mathbb{C}[[x]]$. The module of Kähler differentials $\Omega^1_{S/k}$ is finitely generated. Hence, we can apply Theorem 4.4.7 and obtain: $e_S = e_T$. Since $\Omega^1_{T/\mathbb{C}}$ is also finitely generated, we get that $\Omega^1_{T/\mathbb{C}} = \widetilde{\Omega}^1_{T/\mathbb{C}}$. Therefore, we are able to apply Theorem 4.4.6 to get: $e_T = e_W$. By Lemma 3.3.8 and Corollary 3.3.12, we can derive that the universally finite module $\widetilde{\Omega}^1_{(\mathbb{C}\{x\}/I\mathbb{C}\{x\})/\mathbb{C}}$ exists. So again applying Theorem 4.4.6 yields $e_{\mathbb{C}\{x\}/I\mathbb{C}\{x\})} = e_W$ and this leads to the desired equality. ■

The examples considered here make use of some results, we mentioned in the introduction. We give a short overview and we use the SINGULAR implementation of the algorithms of Section 5.4. For details, see Chapter B.

Theorem 5.5.4. *Let* R *be a reduced Gorenstein curve singularity. Then* $c_R = 2\delta_R$.

Proof. See [Bas63, Corollary 6.5]. ■

Theorem 5.5.5. *Let* R *be a reduced curve singularity. Then we have the inequalities:*

$$e_R \leq \mu_R + 2\delta_R - c_R \leq 3\delta_R - r_R,$$

where μ_R *denotes the Milnor number of* R *and* r_R *the number of branches. In particular, taking Theorem 5.5.4 into account: If* R *is also Gorenstein, then we have:* $e_R \leq \mu_R$.

Proof. See [Gre82, Theorem 2.5]. ■

Theorem 5.5.6. *Let* R *denote a reduced Gorenstein curve singularity. Then* R *is quasi-homogeneous if and only if* $e_R = \mu_R$.

Proof. See [GMP85, Satz 2.1]. ■

Theorem 5.5.7. *Let* R *be a reduced curve singularity. Then we have:* $\mu_R = 2\delta_R - r_R + 1$

Proof. See [BG80, Proposition 1.2.1]. ■

We start with examples of plane curves:

Example 5.5.8.

a) Let $f = x^2 - y^3$ and $R = \mathbb{C}\{x,y\}/\langle f\rangle$. Then f is clearly quasi-homogeneous with weights $(3,2)$. Since f is reduced and Gorenstein, in fact, it is even a complete intersection, we expect from the above theorems that $e_R = \mu_R$ and $c_R = 2\delta_R$. A SINGULAR computation shows that this is true:

```
ideal I = x2-y3;
curveDeltaInv(I);
1
curveConductorMult(I);
2
curveDeligneNumber(I);
2
milnor(I);
2
```

From Theorem 5.5.7, we can also deduce that $r_R = 1$. So, as expected, the curve is irreducible.

b) Another quasi-homogeneous plane curve with two branches is given by $R = \mathbb{C}\{x,y\}/\langle f\rangle$, where $f = x^2 - y^4 = (x-y^2)(x+y^2)$. The curve is reduced and Gorenstein, so we expect the same results as above. If we do a similar SINGULAR computation, we get: $\delta_R = 2$, $c_R = 4$, $\mu_R = 3$ and $e_R = 3$. The number of branches is $r_R = 2$.

c) Now let us consider a curve which is not quasi-homogeneous, namely $f = x^4 + x^2y^2 - x^2y^3 - y^5$ and $R = \mathbb{C}\{x,y\}/\langle f\rangle$. We can use SINGULAR to check quasi-homogeneity with Saito's criterion that we mentioned in the introduction: f is quasi-homogeneous if and only if f lies in the ideal that is generated by its partial derivatives, see [Sai71].

```
poly f = x4+x2y2-x2y3-y5;
ideal J = jacob(f);
reduce(f,std(J));
1/4y5
```

Hence, f is not quasi-homogeneous and we expect from Theorem 5.5.6 and 5.5.5 that $e_R < \mu_R$. From a SINGULAR session, we obtain: $\delta_R = 6$, $c_R = 12$, $\mu_R = 10$ and $e_R = 9$.
In the ring $\mathbb{Q}[x,y]_{\langle x,y\rangle}$, we can factorize f into the two irreducible factors $x^2 + y^2$ and $x^2 - y^3$. Therefore, the number of branches of $\mathbb{Q}[x,y]_{\langle x,y\rangle}/\langle f\rangle$ is 2. But by Theorem 5.5.7, we get that $r_R = 3$. So in general, the number of analytic branches exceeds the number of branches over the polynomial ring. This is due to the fact that roots can be expressed in terms of convergent power series.

The following examples use more than 2 variables and their computation takes more time:

Example 5.5.9.

a) Let $I = \langle x^2+y^2+z^2, x^2y-y^2z\rangle \trianglelefteq \mathbb{C}\{x,y,z\}$ and set $R = \mathbb{C}\{x,y,z\}/I$. Then R is a reduced complete intersection and I is homogeneous. Note that we can use the SINGULAR command `is_ci` to test if R is a complete intersection. For the invariants, we get the following: $\delta_R = 9$, $c_R = 18$, $\mu_R = 13$ and $e_R = 13$. The number of branches is $r_R = 6$.

b) Another example for a complete intersection is $R = \mathbb{C}\{x,y,z\}/I$, where I is given by $I = \langle xy - z^2, zx - y^2\rangle$. We get: $\delta_R = 4$, $c_R = 8$, $\mu_R = 5$ and $e_R = 5$. The number of branches is 4.

c) Consider $R = \mathbb{C}\{x,y,z\}/I$, where $I = \langle x^2-xz+x^2z-xz^2, xy-yz, y^2+xz+xz^2\rangle$. This is a reduced curve singularity of Cohen-Macaulay type $t_R = 2$. This can be computed using the SINGULAR command `CMtype`. Hence, it is not a Gorenstein curve and our computations yield: $\delta_R = 2$, $c_R = 3$, $e_R = 3$. We cannot compute the Milnor number since the command `milnor` only works for complete intersections.
First, we see that $c_R \neq 2\delta_R$. Hence, the Gorenstein assumption of Theorem 5.5.4 is necessary.
Second, we can give bounds for the Milnor number: from Theorem 5.5.5, we can deduce:

$$\begin{aligned} &3 \leq \mu_R + 2\cdot 2 - 3 < 3\cdot 2 \\ \Leftrightarrow\ &2 \leq \mu_R \leq 4 \end{aligned}$$

Taking Theorem 5.5.7 into account, we get that $\mu_R = 5 - r_R$. But since the number of minimal primes of $Q[x]_{\langle x\rangle}/I$ is 2, we get that $r_R \geq 2$. Therefore,

$$\mu_R = 5 - r_R \leq 3$$

and, hence, the Milnor number can be 2 or 3.
The inequality $e_R \leq \mu_R + t_R - 1$ holds for all values of μ_R. This inequality is part of a conjecture in [GMP85]. For details, see Section 5.6.

5.6 Outlook

With the algorithms we are able to compute the delta invariant, the multiplicity of the conductor and the Deligne number for curve singularities. A next step would be to find more algorithms for other invariants. If we could compute the number of branches r, then it would be easy to find the Milnor number for reduced curves by the formula: $\mu = 2\delta - r + 1$, see Theorem

5.5.7. We could then establish a test for quasi-homogeneity for Gorenstein curves. Taking the Cohen-Macaulay type t into account, we could do tests to prove or disprove the following conjecture: for a smoothable curve singularity, we have: $e \leq \mu + t - 1$ and equality holds if and only if the singularity is quasi-homogeneous, see [GMP85].

So far, the Singular algorithms for computing the invariants, especially the algorithm for computing the Deligne number, depends on many time consuming computations: it involves many syzygy-computations, the normalization algorithm and the procedure calls `division`. These algorithms, especially the syzygy-computations are expensive and time-intensive. They occur since we need to compute generators of the I-preserving derivation modules. It would be worth to reduce their number to a minimum and avoid them if possible. So, an optimization of the algorithms would be desirable.

A. Basic facts from commutative algebra

This chapter lists some facts from commutative algebra that were used during the thesis in several chapters. The collected results treat different topics: localization, contraction and extension of ideals, flatness, Artinian rings, product rings and algebras essentially of finite type.

A.1 Extension and contraction of ideals, localization

If we have a ring homomorphism, there is an easy way to extend ideals from one ring to another or to contract ideals:

Definition A.1.1. Let $R \overset{\varphi}{\to} S$ be a homomorphism between rings, I an ideal in R and J an ideal in S.

- The **extension** of I is the ideal $I^e = \langle \varphi(I) \rangle_S \trianglelefteq S$.
- The **contraction** of J is the ideal $J^c = \varphi^{-1}(J) \trianglelefteq R$.

Without further assumptions is not in general true that extending and contracting yields the original ideal. But we can state two inclusions:

Remark A.1.2. Let φ be a homomorphism between rings R and S, I an ideal in R and J an ideal in S. Then we have:

- $I \subseteq I^{ec}$
- $J^{ce} \subseteq J$

Proof. We have a sequence of containments:

$$I \subseteq \varphi^{-1}(\varphi(I)) \subseteq \varphi^{-1}(I^e) = I^{ec}$$

and since $\varphi(\varphi^{-1}(J)) \subseteq J$, we get:

$$J^{ce} = \langle \varphi(J^c) \rangle = \langle \varphi(\varphi^{-1}(J)) \rangle \subseteq J$$

■

The next lemma is well-known and establishes a link between the support of a module and the annihilator:

Lemma A.1.3. *For a ring R and a finitely generated R-module M, we have:* $\mathrm{Supp}(M) = V(\mathrm{Ann}_R(M))$.

Proof. The result is shown in [Stacks, Tag 00L2] ■

If we localize at a multiplicatively closed set that intersects with the annihilator, we get 0:

Lemma A.1.4. *Let R be any ring and M an R-module. If $S \subseteq R$ is a multiplicatively closed set so that $S \cap \mathrm{Ann}_R(M) \neq \emptyset$, then $S^{-1}M = 0$.*

Proof. Let $\frac{m}{s} \in S^{-1}M$ be an arbitrary element and $t \in S \cap \mathrm{Ann}_R(M)$. Then $\frac{m}{s} = \frac{tm}{ts} = 0$. ■

We mentioned before Remark A.1.2 that we need further assumptions to get the equality $J^{ce} = J$. If we contract and extend an ideal from a localization, the relation actually holds.

Lemma A.1.5. *Let R be a ring, W a multiplicatively closed subset of R and J an ideal of $W^{-1}R$. Let $\varphi : R \to W^{-1}R$ be the natural map. Then we have:*

$$J^{ce} = J$$

Proof. By Remark A.1.2. we only have to show the inclusion "$\supseteq$". Therefore, let $x = \frac{a}{b} \in J$. Then we get that $\frac{a}{1} = bx \in J$ and thus: $a \in \varphi^{-1}(J) = J^c$. Hence, $\frac{a}{1} \in J^{ce}$ and finally: $\frac{a}{b} = \frac{1}{b} \cdot \frac{a}{1} \in J^{ce}$. ■

A.2 Flatness

The next two general results about flat modules and flat ring maps are very useful. In fact, Proposition A.2.2 is one of the main reasons why completion and field extension are compatible with the invariants from Chapter 4.

Lemma A.2.1. *If R is a ring, M an R-module, N a submodule of M and F a flat module over R, then:*

$$M/N \otimes_R F = (M \otimes_R F)/(N \otimes_R F)$$

In particular: if S is an R-algebra and I an ideal of S, then $S/I \otimes_R F = (S \otimes_R F)/I(S \otimes_R F)$

Proof. The sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ is exact. If we take the tensor product with F, we obtain the exact sequence: $0 \longrightarrow N \otimes_R F \longrightarrow M \otimes_R F \longrightarrow M/N \otimes_R F \longrightarrow 0$. The claim then follows by the homomorphism theorem. ■

Proposition A.2.2. *Let $R \to S$ be a flat ring homomorphism, M and N R-modules and M finitely presented. Then there is a natural isomorphism of S-modules:*

$$\mathrm{Hom}_R(M, N) \otimes_R S \cong \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

Proof. See [Eis95, Proposition 2.10]. ■

As an example for a faithfully flat extension, we can consider any field extension:

Lemma A.2.3. *Let k be a field and L/k be a field extension. Then L is faithfully flat over k:*

Proof. We have to show that a sequence of k-vector spaces $M \xrightarrow{f} N \xrightarrow{g} P$ is exact if and only if $M \otimes_k L \xrightarrow{f \otimes id} N \otimes_k L \xrightarrow{g \otimes id} P \otimes_k L$ is exact.
We know that L admits a k-basis. Let $\{e_i \mid i \in I\}$ denote this basis, then we can write L as $L = \bigoplus_{i \in I} ke_i$. Using that the tensor product commutes with direct sums, the sequence $M \otimes_k L \xrightarrow{f \otimes id} N \otimes_k L \xrightarrow{g \otimes id} P \otimes_k L$ can be written as:

$$\bigoplus_{i \in I} M \xrightarrow{f \otimes id} \bigoplus_{i \in I} N \xrightarrow{g \otimes id} \bigoplus_{i \in I} P$$

The maps $f \otimes id$ and $g \otimes id$ are then just componentwise applications: $(f)_{i \in I}$ and $(g)_{i \in I}$. Hence, we obtain the equalities: $\mathrm{Im}(f \otimes id) = \bigoplus_{i \in I} \mathrm{Im}(f)$ and $\mathrm{Ker}(g \otimes id) = \bigoplus_{i \in I} \mathrm{Ker}(g)$ and these two are equal if and only if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. ■

As a consequence we can derive: if S is a k-algebra and L/k a field extension. Then $S \to S \otimes_k L$ is flat and if we also take Proposition A.2.2 into account, we can even say more:

Corollary A.2.4. *Let L/k be a field extension and S an k-algebra. Then $S \to S \otimes_k L$ is flat and for S-modules M and N, where M is finitely presented, we have:*

$$\mathrm{Hom}_{S \otimes_k L}(M \otimes_k L, N \otimes_k L) \cong \mathrm{Hom}_S(M, N) \otimes_k L$$

Proof. For any S-module N, we have $N \otimes_k L = N \otimes_S (S \otimes_k L)$. Hence, the flatness of $k \to L$ immediately implies the flatness of $S \to S \otimes_k L$.
The same argument allows us to apply Proposition A.2.2:

$$\begin{aligned}
\mathrm{Hom}_{S \otimes_k L}(M \otimes_k L, N \otimes_k L) &= \mathrm{Hom}_{S \otimes_S (S \otimes_k L)}(M \otimes_S (S \otimes_k L), N \otimes_S (S \otimes_k L)) \\
&\cong \mathrm{Hom}_S(M, N) \otimes_S (S \otimes_k L) \\
&= \mathrm{Hom}_S(M, N) \otimes_k L
\end{aligned}$$ ■

Flatness also influences the Krull dimension of a ring extension:

Proposition A.2.5. *Let $R \to S$ be a flat local homomorphism of local rings. If $\mathfrak{m}$ denotes the maximal ideal of R, then we have:* $\dim S = \dim R + \dim S/\mathfrak{m}S$.

Proof. See [BH93, Theorem A.11]. ■

A.3 Artinian rings and product rings

The only fact that we need about Artinian rings is a standard result: they are all semi-local:

Proposition A.3.1. *Artinian rings have only finitely many maximal ideals.*

Proof. See [AM69, Proposition 8.3]. ■

Now we focus on product rings. The first thing we consider is the behaviour of the Jacobson radical. As one expects, the Jacobson radical of a product is a product itself:

Lemma A.3.2. *Let R, S be rings and let $\mathfrak{J}(R)$, $\mathfrak{J}(S)$ denote the Jacobson radical of R and S respectively. Then we have:*

$$\mathfrak{J}(R \times S) = \mathfrak{J}(R) \times \mathfrak{J}(S)$$

Proof. Let $(r, s) \in \mathfrak{J}(R \times S)$. Then we have for any maximal ideal $\mathfrak{m}_R$ of R: $\mathfrak{m}_R \times S$ is a maximal ideal of $R \times S$. Hence, $(r, s) \in \mathfrak{m}_R \times S$ and therefore, $r \in \mathfrak{m}_R$. Thus, r lies in the intersection of all maximal ideals of R, in $\mathfrak{J}(R)$. Similarly one can show that $s \in \mathfrak{J}(S)$ and altogether: $(r, s) \in \mathfrak{J}(R) \times \mathfrak{J}(S)$. For the converse inclusion, let $(r, s) \in \mathfrak{J}(R) \times \mathfrak{J}(S)$. Let $\mathfrak{m}$ be a maximal ideal of $R \times S$. Then we have two cases:

1. $\mathfrak{m} = \mathfrak{m}_R \times S$, where $\mathfrak{m}_R$ is a maximal ideal of R. Then $r \in \mathfrak{m}_R$ and $(r, s) \in \mathfrak{m}$.

2. $\mathfrak{m} = R \times \mathfrak{m}_S$, where $\mathfrak{m}_S$ is a maximal ideal of S. This is proven similarly.

Hence, $(r, s) \in \mathfrak{J}(R \times S)$. ■

If we have a product ring which is local, it cannot have more than one factor:

Lemma A.3.3. *If R is a local ring of the form $R = S_1 \times \cdots \times S_r$, where the S_i are rings, then $r = 1$.*

Proof. This is [AK70, Chapter VII, Lemma 2.8]. ■

If R and S are rings, M_R, N_R are R-modules and M_S, N_S are S-modules, then $M_R \times M_S$ and $N_R \times N_S$ are $R \times S$-modules by componentwise R- and S-multiplication. The next lemma shows that the Hom functor behaves well under taking products: the homomorphism module of products is the product of the homomorphism modules.

Lemma A.3.4. *Let R, S be rings, M_R, N_R R-modules and M_S, N_S S-modules, then:*

$$\mathrm{Hom}_R(M_R, N_R) \times \mathrm{Hom}_S(M_S, N_S) \cong \mathrm{Hom}_{R\times S}(M_R \times M_S, N_R \times N_S)$$

as $R \times S$-modules.

Proof. To establish an isomorphism between the two modules, define

$$\begin{aligned} \phi : \mathrm{Hom}_R(M_R, N_R) \times \mathrm{Hom}_S(M_S, N_S) &\to \mathrm{Hom}_{R\times S}(M_R \times M_S, N_R \times N_S) \\ (\varphi_R, \varphi_S) &\mapsto \varphi_{R\times S}, \end{aligned}$$

where $\varphi_{R\times S} : M_R \times M_S \to N_R \times N_S$ maps (a, b) to $(\varphi_R(a), \varphi_S(b))$. Then:

- ϕ is well defined since the $\varphi_{R\times S}$ are $R \times S$-linear maps. Let $(r, s) \in R \times S$ and $(a, b) \in M_R \times M_S$, then using that φ_R is R-linear and φ_S is S-linear gives:

$$\begin{aligned} \varphi_{R\times S}((r, s) \cdot (a, b)) &= \varphi_{R\times S}((ra, sb)) \\ &= (\varphi_R(ra), \varphi_S(sb)) \\ &= (r \cdot \varphi_R(a), s \cdot \varphi_S(b)) \\ &= (r, s) \cdot \varphi_{R\times S}(a, b) \end{aligned}$$

- ϕ is $R \times S$-linear: Let $(r, s) \in R \times S$ and $\varphi_R \in \mathrm{Hom}_R(M_R, N_R)$, $\varphi_S \in \mathrm{Hom}_S(M_S, N_S)$. For $(a, b) \in M_R \times M_S$ we obtain:

$$\begin{aligned} \phi((r, s) \cdot (\varphi_R, \varphi_S))(a, b) &= \phi((r\varphi_R, s\varphi_S))(a, b) \\ &= (r\varphi_R(a), s\varphi_S(b)) \\ &= (r, s) \cdot (\varphi_R(a), \varphi_S(b)) \\ &= (r, s) \cdot \phi((\varphi_R, \varphi_S))(a, b) \end{aligned}$$

- ϕ is injective: If $\phi(\varphi_R, \varphi_S)$ is the zero map, it holds for $a \in M_R$ and $b \in M_S$:

$$(0, 0) = \phi((\varphi_R, \varphi_S))(a, b) = (\varphi_R(a), \varphi_S(b))$$

  So φ_R and φ_S were both zero before.

- ϕ is surjective: we construct a preimage for an $R \times S$-linear homomorphism φ in $\mathrm{Hom}_{R\times S}(M_R \times M_S, N_R \times N_S)$. Therefore let

$$\begin{aligned} e_R : M_R &\to M_R \times M_S \\ m &\mapsto (m, 0) \end{aligned} \qquad \begin{aligned} e_S : M_S &\to M_R \times M_S \\ m' &\mapsto (0, m') \end{aligned}$$

be the canonical embeddings and

$$\begin{aligned} \pi_R : N_R \times N_S &\to N_R \\ (n, n') &\mapsto n \end{aligned} \qquad \begin{aligned} \pi_S : N_R \times N_S &\to N_S \\ (n, n') &\mapsto n' \end{aligned}$$

the canonical projections.

For the projection π_R it holds

$$\pi_R((r,s)\cdot(n,n')) = \pi_R(rn, sn') = rn = r \cdot \pi_R(n,n') \tag{A.1}$$

for $r \in R$, $s \in S$, $n \in N_R$, $n' \in N_S$. As a consequence:

$$\begin{aligned} \pi_R(\varphi(a,b)) &= 1_R \cdot \pi_R(\varphi(a,b)) \\ &= \pi_R((1_R, 0)\varphi(a,b)) = \pi_R(\varphi(a,0)) = \pi_R(\varphi(e_R(a))), \end{aligned} \tag{A.2}$$

where $a \in M_R$, $b \in M_S$. This is also true for π_S and can be shown in exactly the same way.

Now define maps $\varphi_R = \pi_R \circ \varphi \circ e_R : M_R \to N_R$ and φ_S similarly. For $r \in R$ and $a \in M_R$ we obtain:

$$\begin{aligned} \varphi_R(ra) &= \pi_R(\varphi(e_R(ra))) \\ &= \pi_R(\varphi((ra, 0))) \\ &= \pi_R((r,0)\cdot\varphi(a,0)) \\ &\overset{(A.1)}{=} r \cdot \pi_R(\varphi(a,0)) \\ &= r \cdot \varphi_R(a) \end{aligned}$$

So φ_R is R-linear and thus an element of $\mathrm{Hom}_R(M_R, N_R)$. And it can be shown in an analogous way that also φ_S is S-linear. Finally, (φ_R, φ_S) is the preimage of φ:

$$\varphi((a,b)) = (\pi_R(\varphi(a,b)), \pi_S(\varphi(a,b)) \overset{(A.2)}{=} (\varphi_R(a), \varphi_S(b))$$

And this implies that $\varphi = \phi(\varphi_R, \varphi_S)$. ■

A.4 Algebras essentially of finite type

The last consideration in this chapter treats algebras essentially of finite type. The class of this algebras is "closed" under passing to algebras of finite type and under base-change:

Lemma A.4.1. *Let R be a ring and T an R-algebra essentially of finite type.*

a) If U is an algebra of finite type over T, then U is essentially of finite type over R

b) If U is any R-algebra, then $T \otimes_R U$ is essentially of finite type over U.

Proof. We start with part a): T is essentially of finite type. Hence, there is an R-algebra of finite type, S, and a multiplicatively closed set W in S so that $T = W^{-1}S$. We have the maps:

$$S \xrightarrow{\psi} W^{-1}S = T \xrightarrow{\varphi} U$$

Since U is of finite type over T, we may replace S by the algebra $S[\underline{x}]$, which is still of finite type over R. Therefore, we can assume, that φ is surjective. Now set $I = \mathrm{Ker}(\varphi)$ and $J = \mathrm{Ker}(\varphi \circ \psi)$, then we have: $I = W^{-1}J$:

If $\frac{a}{b} \in I$, then also $\frac{a}{1} = \frac{b}{1} \cdot \frac{a}{b} \in I$. Hence, $\varphi(\psi(a)) = \varphi(\frac{a}{1}) = 0$ and therefore: $a \in J$. This implies that $\frac{a}{b} = \frac{1}{b} \cdot \frac{a}{1} \in W^{-1}J$. For the other inclusion, let $\frac{a}{b} \in W^{-1}J$, then $a \in J$ and $0 = \varphi(\psi(a)) = \varphi(\frac{a}{1})$. Hence, $\frac{a}{1} \in I$ and finally $\frac{a}{b} \in I$.

Now we can derive: $U = T/I = W^{-1}S/W^{-1}J = \overline{W}^{-1}(S/J)$, where $\overline{W}$ is the image of W under the map $S \to S/J$. Since S/J is of finite type over R, the algebra U is essentially of finite type over R.

To prove part b), we may again write $T = W^{-1}S$, where S is again an R-algebra of finite type. Then S is of the form: $R[\underline{x}]/I$ and we can deduce:

$$S \otimes_R U = U[\underline{x}]/IU[\underline{x}]$$

Hence, $S \otimes_R U$ is of finite type over U.
The map $S \otimes_R U \to W^{-1}S \otimes_R U$ maps the multiplicatively closed set $W \otimes 1$ to units. So we obtain a map from the universal property of localization:

$$\begin{aligned}(W \otimes 1)^{-1}(S \otimes_R U) &\longrightarrow W^{-1}S \otimes_R U \\ \frac{a}{b} &\longmapsto ab^{-1}\end{aligned}$$

This is actually surjective, since we can write any pure tensor as a product: $\frac{e}{f} \otimes u = (e \otimes u) \cdot (f \otimes 1)^{-1}$. Thus, $W^{-1}S \otimes_R U = T \otimes_R U$ is a quotient of an algebra essentially of finite type over U. Now an application of part a) proves the claim. ∎

B. Implementation of algorithms for computing invariants

In this chapter, we present the implementation of the algorithms of Section 5.4 in Singular. We may always assume that we start with a radical ideal in a local polynomial ring $\mathbb{Q}[x]_{\langle x \rangle}$. In Section 5.5, we have shown that the computation of invariants in $\mathbb{Q}[x]_{\langle x \rangle}$ is sufficient to obtain invariants of curve singularities. Hence, these algorithms can be used to compute invariants of curve singularities.

```
///////////////////////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////////////////////
version="version curvesing.lib 1.0.0.5 Aug_2015";
category="Algebraic geometry";
info="
LIBRARY:    curvesing.lib A library for computing invariants of curve singularities
AUTHOR:     Peter Chini, chini@rhrk.uni-kl.de

OVERVIEW:
This library provides a collection of procedures for computing invariants
of curve singularities. Invariants that can be computed are:
    - the delta invariant
    - the multiplicity of the conductor: the colength of the conductor ideal in the
      normalization
    - the Deligne number
    - the colength of derivations along the normalization: the colength of
      derivations relative to derivations on the normalization

In addition, it is possible to compute the conductor of the basering mod I,
where I is an ideal.

THEORY: P. Chini, Computing the Deligne number of curve singularities and an algorithmic
        framework for differential algebras in SINGULAR, 2015

KEYWORDS:   curve singularity;invariants;deligne number

PROCEDURES:
    curveDeltaInv(I);   delta invariant of curve singularity defined by I

    normalConductor(I);     conductor of basering mod I as ideal in the basering
    curveConductorMult(I);  colength of conductor in normalization of curve sing. defined by I

    curveDeligneNumber(I);          Deligne number of curve sing. defined by I
```

```
    curveColengthDerivations(I);    colength of derivations of curve sing. defined by I
";
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////


///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////
//                              Initialization of library                    //
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////


static proc mod_init()
{

    // Libraries needed
    LIB "homolog.lib";
    LIB "normal.lib";

}


///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////
//                              Computation of invariants                    //
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////


///////////////////////////////////////////////////////////////////////////////
//-------------------------------- Delta invariant ----------------------------//
///////////////////////////////////////////////////////////////////////////////


proc curveDeltaInv(ideal I, list #)
"USAGE:      curveDeltaInv(I); I ideal
            curveDeltaInv(I,L); I ideal, L = normal(I,"useRing","prim","wd")
ASSUME:     - I is a radical ideal and defines a curve: basering mod I
            - the basering is a local ring
            - the basefield is perfect
RETURN:     the delta invariant of the curve singularity defined by I
NOTE:       output -1 means: delta invariant is infinite
KEYWORDS:   delta invariant; normalization
SEE ALSO:   curveConductorMult; curveDeligneNumber
EXAMPLE:    example curveDeltaInv; shows an example"
{

    if(size(#) > 0){
        list norma = #;
    }else{
        // Compute the normalization with delta invariants
        list norma = normal(I,"useRing","prim","wd");
    }

    // Pick the total delta invariant
        int delt = norma[3][2];
    return(delt);

}
example
{
```

```
"EXAMPLE:"; echo = 2;
ring R = 0,(x,y,z),ds;

/////////////////////////////
// Finite delta invariant //
/////////////////////////////

ideal I = x2y-y2z,x2-y2+z2;
curveDeltaInv(radical(I));

///////////////////////////////
// Infinite delta invariant //
///////////////////////////////

ideal J = xyz;
curveDeltaInv(radical(J));
}


////////////////////////////////////////////////////////////////////////////////////////////////////
//-------------------------------- Conductor and multiplicity ---------------------------------//
////////////////////////////////////////////////////////////////////////////////////////////////////


proc normalConductor(ideal I, list #)
"USAGE:      normalConductor(I); I ideal
             normalConductor(I,L); I ideal, L = normal(I,"useRing","prim","wd")
ASSUME:      I is a radical ideal
RETURN:      conductor of basering mod I as ideal in the basering
REMARKS:     The procedures makes use of the minimal primes and
             the generators of the normalization given by the normalization algorithm.
KEYWORDS:    conductor; normalization
SEE ALSO:    curveConductorMult; normal
EXAMPLE:     example normalConductor; shows an example"
{

    if(size(#) > 0){
        list norma = #;
    }else{
        // Compute the normalization with delta invariants
        list norma = normal(I,"useRing","prim","wd");
    }

    // Prepare computation
        int r = size(norma[1]);
        int i;
        def savering = basering;
        list min_prime;
    list DEN;
    def S;

    // List of all minimal primes
    for(i = 1; i <= r; i++){
        S = norma[1][i];
        min_prime[i] = conductorMinPrime(S);
    }

        // List of ideals U generating Norm(R/P_i)
        list U = norma[2];

        // List of denominators DEN
    r = size(U);
```

```
        for(i = 1; i <= r; i++){
                DEN[i] = U[i][size(U[i])];
        }

        // Compute the conductor C
        ideal C = 1;
        ideal C_current;
        ideal min_prime_isect;

        for(i = 1; i <= r; i++){

                // Intersection of min_prime_j, j!=i
                min_prime_isect = conductorIdealIntersect(min_prime,i);

                // Compute the quotient
                C_current = std(quotient(min_prime[i] + DEN[i]*min_prime_isect, U[i]));

                // Intersect it with previous computation
                C = intersect(C,C_current);
        }

        return(std(C));

}
example
{
"EXAMPLE:"; echo = 2;

///////////////////////////////////////////
// Computation of small conductor ideals //
///////////////////////////////////////////

ring R = 0,(x,y,z),ds;
ideal I = x2y2 - z;
normalConductor(I);
// The conductor is the whole ring - so the ring is normal
// We can also see this using the delta invariant:
curveDeltaInv(I);

ring S = 0,(a,b,c),dp;
ideal J = abc;
normalConductor(J);
// The conductor is not the whole ring - so it is not normal
// We can also see this using the delta invariant, which is even infinite
curveDeltaInv(J);

kill R,S;

//////////////////////////////////////
// Computation of a bigger example //
//////////////////////////////////////

ring R = 0,(x,y,z,t),ds;
ideal I = xyz - yzt, x2y3 - z2t4;
I = std(radical(I));
// Ideal I
I;
// Conductor
normalConductor(I);
}
```

```
////////////////////////////////////////////////////////////////////////////////


static proc conductorMinPrime(def S)
"USAGE:     conductorMinPrime(S); S ring
ASSUME:     S is a polynomial ring with ideal norid and S/norid is the normalization of
            basering mod P, where P is a minimal prime of I
RETURN:     the ideal P
REMARKS:    The algorithm intersects norid with the basering.
NOTE:       the algorithm is for interior use only. We apply it to avoid a second computation
            of the minimal primes
KEYWORDS:   minimal primes; normalization
SEE ALSO:   conductorIdealIntersect"
{

        // Variables of basering as product
        int n = nvars(basering);
    int i;
        poly var_base = 1;
        for(i = 1; i <= n; i++){
                var_base = var_base*var(i);
        }

    // Switch to normalization
    def savering = basering;
    setring S;
        poly var_base = imap(savering,var_base);

        // Variables of S as product
        poly var_ext = 1;
        n = nvars(basering);
        for(i = 1; i <= n ; i++){
                var_ext = var_ext*var(i);
        }

        // Variables to eliminate
        poly var_elim = var_ext/var_base;

        // Compute norid intersect basering = minimal prime
        ideal min_prime = eliminate(norid,var_elim);

    // Switch to R and return
    setring savering;
    ideal min_prime = imap(S,min_prime);
    return(min_prime);

}


////////////////////////////////////////////////////////////////////////////////


static proc conductorIdealIntersect(list id, int miss)
"USAGE:     conductorIdealIntersect(id,miss); id list, miss int
ASSUME:     id is a list of ideals
RETURN:     the intersection of all ideals in id except the one chosen via miss
NOTE:       - the index can be chosen outside the list
            - the empty intersection is the whole ring
KEYWORDS:   intersection"
{

        int n = size(id);
```

```
        ideal in_sect = 1;
        int i;

        // Intersect ideals
        for(i = 1; i <= n; i++){
                if(i != miss){
            in_sect = intersect(in_sect,id[i]);
                }
        }

        return(in_sect);

}


////////////////////////////////////////////////////////////////////////////////////////////////


proc curveConductorMult(ideal I, list #)
"USAGE:     curveConductorMult(I); I ideal
            curveConductorMult(I,L); I ideal, L = normal(I,"useRing","prim","wd")
ASSUME:     - I is a radical ideal and defines a curve: basering mod I
            - the basering is a local ring
            - the basefield is perfect
RETURN:     the multiplicity of the conductor of the curve singularity defined by I:
            the colength of the conductor in the normalization
KEYWORDS:   conductor; multiplicity
SEE ALSO:   normalConductor
EXAMPLE:    example curveConductorMult; shows an example"
{

    if(size(#) > 0){
        list norma = #;
    }else{
        // Compute the normalization with delta invariants
        list norma = normal(I,"useRing","prim","wd");
    }

    // delta invariant
    int delta = curveDeltaInv(I,norma);
        // If the delta invariant is infinite, the conductor multiplicity is as well
        if(delta == -1){
                return(-1);
        }

    // Conductor
    ideal C = normalConductor(I,norma);
    int c_dim = vdim(std(C));
    if(c_dim == -1){
        return(-1);
    }

        // Return conductor multiplicity
        return(vdim(std(C)) + delta);

}
example
{
"EXAMPLE:"; echo = 2;

/////////////////////////////////////////////
// Mutltiplicity of the conductor of curves //
```

```
//////////////////////////////////////////////

ring R = 0,(x,y,z),ds;

// Example 1:
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveConductorMult(I);

// Example 2:
ideal I = x*(y+z)^3 - y3, x2y2 + z5;
I = std(radical(I));
curveConductorMult(I);

kill R;

////////////////////////////////////////////////////////
// Mutltiplicity of the conductor of Gorenstein curve //
////////////////////////////////////////////////////////

ring R = 0,(x,y),ds;
ideal I = xy;

// In such a case, the conductor multiplicity c satisfies: c = 2*delta
// Delta invariant:
curveDeltaInv(I);
// Conductor Multiplicity:
curveConductorMult(I);
}


////////////////////////////////////////////////////////////////////////////////////////////////
//--------------------------------------- Deligne number ---------------------------------------//
////////////////////////////////////////////////////////////////////////////////////////////////


proc curveDeligneNumber(ideal I, list #)
"USAGE:     curveDeligneNumber(I); I ideal
            curveDeligneNumber(I,L); I ideal, L = normal(I,"useRing","prim","wd")
ASSUME:     - I is a radical ideal and defines a curve: basering mod I
            - the basering is a local ring
            - the basefield has characteristic 0
RETURN:     the Deligne number of the curve singularity defined by I
REMARKS:    The Deligne number e is defined by: e = 3delta - m.
            So the algorithm splits the computation into two parts: one part computes the delta
            invariant, the other part the colength of derivations m.
KEYWORDS:   deligne number; invariant
SEE ALSO:   curveColengthDerivations, curveDeltaInv
EXAMPLE:    example curveDeligneNumber; shows an example"
{

    if(size(#) > 0){
        list norma = #;
    }else{
        // Compute the normalization with delta invariants
        list norma = normal(I,"useRing","prim","wd");
    }

    int delt = curveDeltaInv(I,norma);
    int m = curveColengthDerivations(I,norma);
    return(3*delt - m);

```

```
}
example
{
"EXAMPLE:"; echo = 2;

/////////////////////////////
// Deligne number of curves //
/////////////////////////////

// Example 1:
ring R = 0,(x,y,z),ds;
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveDeligneNumber(I);

// Example 2:
ring S = 0,(x,y),ds;
ideal I = (x+y)*(x2-y3);
curveDeligneNumber(I);

// Example 3:
ideal J = (x2-y3)*(x2+y2)*(x-y);
curveDeligneNumber(J);
// Let us also compute the Milnor number of this complete intersection:
milnor(J);

// We see that the Milnor number is bigger than the Deligne number. Hence, this
// curve cannot be quasi homogeneous. This can also be verified by Saito's criterion:
reduce(J[1],std(jacob(J[1])));
}


///////////////////////////////////////////////////////////////////////////////////////////////


proc curveColengthDerivations(ideal I, list #)
"USAGE:     curveColengthDerivations(I); I ideal
            curveColengthDerivations(I,L); I ideal, L = normal(I,"useRing","prim","wd")
ASSUME:     - I is a radical ideal and defines a curve: basering mod I
            - the basering is a local ring
            - the basefield has characteristic 0
RETURN:     the colength of derivations of the curve singularity defined by I:
            the colength of derivations relative to derivations on the normalization
KEYWORDS:   deligne number; invariants; colength of derivations
SEE ALSO:   curveColengthDerivationsComp, curve Ddim
EXAMPLE:    example curveColengthDerivations; shows an example"
{

    if(size(#) > 0){
        list norma = #;
    }else{
        // Compute the normalization with delta invariants
        list norma = normal(I,"useRing","prim","wd");
    }

    int r = size(norma[1]);
    int i,j;
    ideal U,A,B;
    module Der_P;
    def S;
    def savering = basering;

```

```
    // List of minimal primes and their derivation modules
    list min_prime;
    list der_mod;

    // Colength of derivations of any branch, m_delta and total colength of derivations
    int m_i;
    int m_delta;
    int ext_number;

    // Go through the irreducible components and compute thecolength of derivations m_i
    for(i = 1; i <= r; i++){
        // Derivations preserving the minimal primes
        S = norma[1][i];
        U = norma[2][i];

        min_prime[i] = conductorMinPrime(S);
        der_mod[i] = find_der(min_prime[i]);
        Der_P = der_mod[i];

        // Switch to normalization of R/P and compute colength of derivations
        setring S;
        ideal U = imap(savering,U);
        module Der_P = imap(savering,Der_P);

        m_i = curveColengthDerivationsComp(Der_P,U,norid);

        // Add colength of derivations of this branch to total colength of derivations
        ext_number = ext_number + m_i;
        setring savering;
    }

    // Now compute m_delta via curveDdim
    A = min_prime[1];
    B = std(1);

    for(i = 2; i <= r; i++){
        A = intersect(A,B);
        B = min_prime[i];
        m_delta = m_delta + curveDdim(A,find_der(A),B,find_der(B));
    }

    // Add this to the colength of derivations
    ext_number = ext_number + m_delta;
    return(ext_number);

}
example
{
"EXAMPLE:"; echo = 2;

//////////////////////////////////////
// colength of derivations of curves //
//////////////////////////////////////

// Example 1:
ring R = 0,(x,y,z),ds;
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveColengthDerivations(I);

// Example 2:
ring S = 0,(x,y),ds;
```

```
ideal I = (x+y)*(x2-y3);
curveColengthDerivations(I);

// Example 3:
ideal J = (x2-y3)*(x2+y2)*(x-y);
curveColengthDerivations(J);
}


///////////////////////////////////////////////////////////////////////////////////////////////


static proc curveColengthDerivationsComp(module Der_P, ideal U, ideal relid)
"USAGE:      curveColengthDerivationsComp(Der_P,U,relid); Der_P module, U ideal, relid ideal
ASSUME:      let R denote the localized polynomial ring and I the ideal of the procedure
             curveColengthDerivations, then assume:
             - the basering is the normalization of R/P, where P is a minimal prime ideal of I
             - Der_P is the module of P-preserving derivations
             - U containts the generators of the normalization of R/P
             - relid is the ideal of relations that hold in the normalization of R/P
RETURN:      the colength of derivations of the curve R/P:
             the colength of derivations relative to derivations on the normalization
NOTE:        the procedure is for interior use only - it is part of the computation of
             the total colength of derivations
KEYWORDS:    colength of derivations
SEE ALSO:    curveAdjustModule, curveExtDerModule"
{

    int k;

    // Adjust the generators of Der_P to the new variables T(1),...,T(k),x(1),...,x(n)
    // if there are new variables - check number of blocks
    if(size(ringlist(basering)[3]) >= 3){
        k = size(ringlist(basering)[3][1][2]);
        Der_P = curveAdjustModule(Der_P,k);
    }

    // Extend the derivation module to the normalization
    Der_P = curveExtDerModule(Der_P,U,relid);

    // Derivations preserving the relation ideal
    module Der_relid = find_der(relid);

    // Quotient module with relations given by relid
    Der_P = Der_P + relid*freemodule(nvars(basering));
    module quotient_mod = modulo(Der_relid,Der_P);
    k = vdim(std(quotient_mod));

    if(k == -1){
        ERROR("Colength of derivations not finite !");
    }

    return(k);

}


///////////////////////////////////////////////////////////////////////////////////////////////


static proc curveExtDerModule(module Der_P, ideal U, ideal relid)
"USAGE:      curveExtDerModule(Der_P,U,relid); Der_P module, U ideal, relid ideal
```

```
ASSUME:      let R denote the localized polynomial ring and I the ideal of the procedure
             curveColengthDerivations, then assume:
             - the basering is the normalization of R/P, where P is a minimal prime ideal of I
             - Der_P is the module of P-preserving derivations
             - U containts the generators of the normalization of R/P
             - relid is the ideal of relations that hold in the normalization of R/P
RETURN:      The derivation module lifted to the normalization
REMARKS:     the generators of Der_P are extended via the quotient rule
NOTE:        the procedure is for interior use only - it is part of the computation of
             the total colength of derivations
KEYWORDS:    derivations; extend derivations
SEE ALSO:    curveColengthDerivationsComp"
{

    int k = size(Der_P);
    int n = size(U) - 1;
    int i,j;

    module M_ext;
    vector delt;
    vector delt_ext;

    poly g = (U[n+1])^2;
    poly f;
    poly Un = 1;
    matrix D[k][n];
    matrix G[k][n];
    list temp_div;

    for(i = 1; i <= k; i++){
        delt = Der_P[i];

        // Extend to new variables by quotient rule
        for(j = 1; j <= n; j++){
            f = vecDerivationEval(delt,U[j])*U[n+1] - vecDerivationEval(delt,U[n+1])*U[j];

            // Division
            temp_div = division(f,ideal(g) + relid);

            // Units
            D[i,j] = temp_div[3][1,1];
            // Unit Un is product of all D[i][j]
            Un = Un*D[i,j];

            // Factor of dividing by g
            G[i,j] = temp_div[1][1,1];
        }
    }

    // Extension of the generating derivations
    for(i = 1; i <= k; i++){
        delt_ext = Un*Der_P[i];

        // Now add the images of the new variables multiplied by the units
        for(j = 1; j <= n; j++){
            delt_ext = delt_ext + (Un / D[i,j])*G[i,j]*gen(j);
        }

        M_ext[i] = delt_ext;
    }

    return(M_ext);
```

```
}


////////////////////////////////////////////////////////////////////////////////


static proc curveAdjustModule(module M, int k)
"USAGE:     curveAdjustModule(M,k); M module, k int
RETURN:     the module M with shifted (by k) generators
NOTE:       the procedure is for interior use only - it is part of the computation of
            the total colength of derivations
KEYWORDS:   adjust module
SEE ALSO:   curveColengthDerivationsComp"
{

    module M_copy = M;
    int n = size(M);
    int vs,i,j;
    vector v,w;

    // Adjust dimension of generators
    for(i = 1; i <= n; i++){
        v = M_copy[i];
        vs = nrows(v);

        for(j = 1; j <= vs; j++){
            w = w + v[j]*gen(j+k);
        }

        M[i] = w;
        w = 0;
    }

    return(M);

}


////////////////////////////////////////////////////////////////////////////////


static proc curveDdim(ideal I, module DI, ideal J, module DJ)
"USAGE:     curveDdim(I,DI,J,DJ); I,J ideal, DI,DJ module
ASSUME:     DI are the I-preserving derivations and DJ are the J-preserving derivations
RETURN:     d(I,J) = dim_k (DI + DJ / (I+J)*Der(R))
NOTE:       the procedure is part of the computations of the colength of derivations
KEYWORDS:   derivation module; logarithmic derivations
SEE ALSO:   curveColengthDerivations"
{

    module M = DI+DJ;
    module N = (I+J)*freemodule(nvars(basering));
    module H = modulo(M,N);
    int k = vdim(std(H));

    if(k == -1){
        ERROR("d-dimension not finite !");
    }

    return(k);

```

```
}


////////////////////////////////////////////////////////////////////////////////////////////////


static proc vecDerivationEval(vector delt, poly f)
"USAGE:     vecDerivationEval(delt,f); delt vector, f poly
ASSUME:     delt does not have more rows than the number of variables in the basering
RETURN:     the image of f under delt, if we consider delt as derivation
REMARKS:    We identify derivations as vectors
NOTE:       - the procedure is for interior use only - it is part of the computation of
            the total colength of derivations
            - it is used to apply the quotient rule
KEYWORDS:   derivation
SEE ALSO:   curveExtDerModule"
{

    int n = nrows(delt);
    int i;
    poly eval_;

    for(i = 1; i <= n; i++){
        eval_ = eval_ + delt[i]*diff(f,var(i));
    }

    return(eval_);

}


////////////////////////////////////////////////////////////////////////////////////////////////


static proc find_der(ideal I)
"USAGE: find_der(I); I ideal
RETURN: generators of the module of logarithmic derivations
REMARK: Algorithm by R. Epure, Homogeneity and Derivations on Analytic Algebras, 2015"
{
    // Dummy variables and Initialization:
    int k,i,n,m;

    //generating matrix for syzygie computation:
    n = nvars(basering);
    m = size(I);
    ideal j = jacob(I);

    matrix M=matrix(j,m,n);
    for (i = 1; i <= m; i++){
        M = concat(M,diag(I[i],m));
    }
    module C = syz(M);
    module D;

    for(i = 1; i <= size(C); i++){
        D = D + C[i][1..n];
    }

    //Clearing memory
    kill j;
    kill C;
    kill M;
```

```
    return(D);
}


//////////////////////////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////////////////////////
```

C. A Singular implementation of differential algebras

This chapter is a short introduction to the implementation of differential algebras in SINGULAR. The library containing the source code is called `difform.lib` and it can be used to compute with differential forms, differential algebras and derivations in SINGULAR without leaving the basering.

C.1 Representation and construction of differential algebras

We want to represent universal differential algebras over quotients of polynomial rings in SINGULAR. Therefore, we need a structural result: in Proposition 3.2.13, it was shown that the universal differential algebra is the exterior algebra of the module of Kähler differentials. Since we know how this module looks like in the case, where $R = k[x]/I$, we can deduce as in Example 3.2.15: if $I = \langle f_1, \ldots, f_r \rangle$, then

$$\Omega_{R/k} = \bigwedge \Omega^1_{R/k} = \bigwedge \left(\bigoplus_{i=1}^{n} k[x]dx_i \,/ \langle df_1, \ldots, df_r, f_1, \ldots, f_r \rangle_{k[x]} \right)$$

Hence, we have to find a representation for the exterior algebra of the module $\bigoplus_{i=1}^{n} k[x]dx_i \,/ \langle df_1, \ldots, df_r, f_1, \ldots, f_r \rangle_{k[x]}$. Since this is a non-commutative ring, we represent it as such: it will be the ring with commutative variables $\mathtt{x_1}, \ldots, \mathtt{x_n}$ and non-commutative variables $\mathtt{Dx_1}, \ldots, \mathtt{Dx_n}$ modulo the relations: $(\mathtt{Dx_i})^2 = 0$, $\mathtt{Dx_i Dx_j} = -\mathtt{Dx_j Dx_i}$ for $i \neq j$ and $\mathtt{f_1}, \ldots, \mathtt{f_r}, \mathtt{df_1}, \ldots, \mathtt{df_r}$. If the basering has name `R`, then the differential algebra will be stored with name `Omega_R`. The monomial ordering of the basering variables $\mathtt{x_1}, \ldots, \mathtt{x_n}$ is preserved, the `D`-variables are ordered by `dp`. We make use of a block ordering in `Omega_R`, where the `D`-variables are in the first block and the $\mathtt{x_i}$ are in the second block. We get that elements of $\Omega_{R/k}$, so called differential forms, can be represented as polynomials in `Omega_R`.
The described construction is done by the procedure `diffAlgebraStructure`. It gets called by the user-available procedure `diffAlgebra`. The differential

algebra is assigned to the basering via an attribute. One can always access the name of the differential algebra by calling

`attrib(basering,"diffAlgebra")`

Differential forms are polynomials in `Omega_R`. So computing with them would be possible if we leave the basering. Since this approach is not intuitive, we decided to make differential forms available over `R` itself with two new types:

- The type `difvar` carries as an attribute a string and represents the forms $dx_1, \ldots, dx_n$.
- The type `difform` carries as an attribute a polynomial and represents more general differential forms of $\Omega_{R/k}$

After constructing the differential algebra in the procedure `diffAlgebra`, the procedure `diffAlgebraGens` is called. This generates the forms $\mathtt{dx_1}, \ldots, \mathtt{dx_n}$ as objects of type `difvar`: it creates the objects $\mathtt{dx_i}$ and stores in the string attribute of $\mathtt{dx_i}$ the variable $\mathtt{Dx_i}$. Hence, they are available over the basering since objects of user-defined types are always global. In Section C.2, we will explain how objects of type `difvar` and `difform` are constructed and how we can do computations with them.

Two important structural procedures for computing with differential algebras are `diffAlgebraCheck` and `diffAlgebraSwitch`. The first procedure tests if the differential algebra of the basering `R` has already been constructed. This is done via the attribute `attrib(basering,"diffAlgebra")`. If the differential algebra has not been constructed yet, the attribute is not set. Hence, the procedure aborts with an error. The second procedure is used to change the basering to the differential algebra: it changes the ring from `R` to `Omega_R`. This is for example needed when we define arithmetic operations of objects of type `difform`.
Other available structural procedures for differential algebras are:

- `diffAlgebrachangeOrd`: a procedure which can be called if the current basering is `Omega_R`. It constructs a ring with the structure of the differential algebra but with changed monomial ordering. This is useful, when we want to sort lists of differential forms with respect to a particular ordering as in the print-procedure.
- `diffAlgebraRelations`: returns the relation ideal of the non-commutative quotient ring `Omega_R` as a list of objects of type `difform`.
- `diffAlgebraListGens` returns a list, sorted by the monomial ordering on `Omega_R`, of the generators of the differential algebra as module over the basering: all products

$$dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j},\ 1 \leq \nu_1 < \cdots < \nu_j \leq n$$

If R is the polynomial ring, then the differential algebra $\Omega_{R/k}$ is free and the procedure returns the basis of $\Omega_{R/k}$ as R-module. There is also the option to choose a particular degree. Then the procedure returns the generators of a graded part of $\Omega_{R/k}$.

Further available procedures which concern the structure of the differential algebra `Omega_R` are type casts. They can be found under the caption *Type casts and conversions* in the `difform.lib`.

C.2 Construction and basic operations of differential forms

First, we make precise how we can construct objects of type `difform`. The procedure `difformFromType` constructs differential forms from several input types:

- `difvar`
- `poly`
- `int`
- `bigint`
- `number`
- `vector`

Let `dx_i` be of type `difvar`. In fact, these are the only objects of type `difvar` which can occur in the `difform.lib`. Then the constructor-subroutine `difformFromDifvar` is called. It creates an `difform`-object `df` and sets the polynomial attribute of `df` to `Dx_i`.
A big advantage of this type cast is that we do not need special arithmetic operations for the type `difvar`. We just cast elements of this type directly to differential forms and can therefore apply procedures, written for the type `difform`.
If the input is an object `f` of type `poly`, the subroutine `difformFromPoly` is called. It maps the polynomial `f` to the differential algebra `Omega_R` and stores it in a `difform`-object `df`.
For `vector`-input, we identify the differential algebra $\Omega_{R/k}$ as a quotient of the free module R^{2^n}, see Example 3.2.15. A given vector `v` is converted to an object of type `difform` by the procedure `difformFromVector`. First, a list `GEN_list`, containing the 2^n generators of $\Omega_{R/k}$ as R-module, is computed via `difformListGens`. Then any standard basis vector e_i of R^{2^n} is identified with the i-th generator: `GEN_list[i]`. The resulting differential form is constructed as: $\sum_{i=1}^{2^n} \mathtt{v}[\mathtt{i}] \cdot \mathtt{GEN\_list}[\mathtt{i}]$.

For the other listed types, we also use the constructor `difformFromPoly`.

Now we consider some basic structural procedures which allow us to decompose or print differential forms:

A very important routine for objects of type `difform` is `difformCoef`. Any differential form $\omega \in \Omega_{R/k}$ can be written as a sum:

$$\omega = \sum_{j=0}^{n} \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq n} s_{\nu_1 \ldots \nu_j} dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j} \tag{C.1}$$

The procedure `difformCoef` returns the coefficients $s_{\nu_1 \ldots \nu_j} \in R$ and the corresponding R-module generators $dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$ of $\Omega_{R/k}$ in a list. In fact, we apply the command `coef` in the differential algebra `Omega_R` to get the coefficients. Then we map the results back to `R`.

An application of `difformCoef` is a type cast from `difform` to `string`. This is realized by the procedure `difformToString`. First, the routine applies `difformCoef` to a given differential form `df` to obtain a decomposition into coefficients and generators as in (C.1). Then, the generators get ordered with respect to a monomial ordering, which may be changed. Finally, a string is constructed which contains the ordered generators and the corresponding coefficients. The procedure `difformPrint` makes use of this and allows us to print differential forms.

Another useful procedure concerning the structure of differential forms is `difformIsGen`. It provides a test for differential forms: the procedure returns 1 if the given object of type `difform` is an R-module generator of the differential algebra $\Omega_{R/k}$, a product $dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$. Otherwise, 0 is returned. Such products are identified via `difformCoef`: if the returned list has only one entry and the corresponding coefficient is 1, then the given object already was a generator.

Mapping polynomials from a ring `R` to a ring `S` is usually done using `imap` or `fetch` in SINGULAR. To provide a similar tool for differential forms, we have constructed the procedures `difformImap` and `difformFetch`.

- Like `imap`, the procedure `difformImap` maps variables to variables with equivalent names.
- Like `fetch`, the procedure `difformFetch` maps variables to variables with equivalent positions.

Note that the `D`-variables of $\Omega_{R/k}$ are always mapped to `D`-variables of $\Omega_{S/k}$. The same holds for the ring variables.

The difform.lib provides basic arithmetic operations for differential forms. The idea of those procedures is always the same: since we know that a differential form df carries as content a polynomial g in Omega_R, we change the ring to Omega_R and perform polynomial operations on g there. From the result, we construct a new differential form and return to the ring R.
Supported operations are: addition, subtraction, negation, multiplication, taking powers, division, equality, non-equality, greater and smaller - with respect to a monomial ordering and reduction with respect to the zero ideal. The latter is helpful when we compute over a quotient ring since the SINGULAR-option qringNF does not support the ring-change that we perform.
For most of this arithmetic operations, we have installed shortcuts - this supports the intuitive use of the library. A valid expression would be:

```
ring R = 0,(x,y,z),dp;
diffAlgebra();
difform df = dx + dy*dz + (4x-y2)*dx*dz;
```

Further operations for objects of type difform concern the degree of differential forms. Since $\Omega_{R/k}$ is graded, the degree of a general element $\omega \in \Omega_{R/k}$ is the highest degree of a homogeneous element occurring in ω. The procedure difformDeg determines this degree by applying the SINGULAR command deg to the polynomial content g of a difform-object df with extra weights: we set the weights of the ring variables $\mathtt{x_1}, \ldots, \mathtt{x_n}$ to 0 since they do not contribute to the degree in $\Omega_{R/k}$. The weights of the D-variables is set to 1. Then the command deg returns the desired degree.
There is also a procedure difformIsHomog which tests a differential form to be homogeneous. The idea is similar to above: we use the command homog and apply it to the polynomial content of the given differential form using the same weights as above.
As a combination of both mentioned procedures, we can test if a differential form is homogeneous of a given degree. Therefore, we use difformIsHomogDeg.
It is also possible to compute a homogeneous decomposition of a differential form df]. This is done by the procedure difformHomogDecomp. It computes a decomposition, using difformCoef and since we can test the degree of the generators, we can build the homogeneous decomposition.

The last consideration in this section are two procedures that deal with lists of differential forms:

- The procedure difformListContains uses that we can determine if two differential forms are equal. It returns 1 (or an index) if a given differential form is in a given list of differential forms. Otherwise, it returns 0.

- The procedure `difformListSort` is an implementation of *insertion sort* for lists of differential forms.

C.3 Computations with the universal derivation

This short section explains how the implementation of the universal derivation works. We will always denote the universal derivation by d and consider it as a map between graded parts of the differential algebra:

$$d : \Omega^j_{R/k} \to \Omega^{j+1}_{R/k}$$

For the theory, we refer to Example 3.2.3.
We provide two implementations, one for the polynomial case and one for the general case:

1. The procedure `difformUnivDer` computes the universal derivation of polynomials. The idea is simple: we build all partial derivations $\frac{\partial f}{\partial x_i}$ with the command `diff`, multiply them by the differential forms dx_i and sum them up. We obtain $\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$ and this is the universal derivation applied to f. This was shown in Section 3.1.

2. The procedure `difformDiff` decomposes a given differential form ω by using `difformCoef`. Then we can write ω as

$$\omega = \sum_{j=0}^{n} \sum_{1 \le \nu_1 < \cdots < \nu_j \le n} s_{\nu_1 \dots \nu_j} dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$$

   In Example 3.2.3, it was shown that:

$$d\omega = \sum_{j=0}^{n} \sum_{1 \le \nu_1 < \cdots < \nu_j \le n} ds_{\nu_1 \dots \nu_j} \wedge dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$$

   Therefore, the procedure applies `difformUnivDer` to the polynomials $s_{\nu_1 \dots \nu_j}$, multiplies this by $dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$ and sums everything up. The result is exactly $d\omega$.

For polynomials it is not important which procedure is applied. For input of type `poly`, they have exactly the same output.

C.4 Representation and construction of derivations

Like for the differential algebras and differential forms, we explain in this section how we can represent derivations on a computer. We will make use of the already defined type `difform` and the isomorphism

$$\mathrm{Hom}_R(\Omega^1_{R/k}, R) \cong \mathrm{Der}_k(R)$$

This allows us to identify derivations as maps from $\Omega^1_{R/k}$ to R. Such maps are uniquely determined by their images of the dx_i. Therefore, we introduce a new type `derivation`. Objects of this type contain a list of lists `L`. The ideal is the following: `L[1][i]` is an R-module generator of $\Omega^1_{R/k}$. The image of this generator is stored in `L[2][i]`.

Derivations can be constructed via the procedure `derivationFromType`. It checks the type of the input argument and if this is of type `list`, the subroutine `derivationFromList` is called. This calls a further subroutine, `derivationCheckList`, which checks if the given list, call it `L`, has the right form for being a derivation. It is required that `L` determines exactly one image for any differential form $\mathtt{dx_i}$. If this test is successful, the procedure `derivationFromList` creates an object `phi` of type `derivation`, sorts `L` with respect to the monomial ordering on `Omega_R` and stores `L` in the attribute of `phi`.

To print derivations in a particular nice way, we provide the procedure `derivationToString` that performs a type cast from `derivation` to `string`. Since derivations contain differential forms, we make use of the procedure `difformToString` to cast them. To get the direct print, one can also apply `derivationPrint`.

There are basic arithmetic operations of derivations available: addition, subtraction, negation, equality, non-equality, evaluation and multiplication by polynomials, which models the module structure of the derivation module. Except for the evaluation-procedure, the idea is always the same. We explain it in the case of addition: the procedure `derivationAdd` takes as input two derivations φ and γ. These carry two lists `L` and `T`. The sum $\varphi+\gamma$ should map the dx_i to $\varphi(dx_i)+\gamma(dx_i)$. These images are stored in the lists `L` and `T` and since they are sorted the same way (by the constructor), we can just do an addition of the polynomials $\mathtt{L[i]} + \mathtt{T[i]}$ to obtain $\varphi(dx_i)+\gamma(dx_i)$. This is done for any dx_i and we get the sum of the two derivations.
As for differential forms, shortcuts were added to support an intuitive handling.

The procedure `derivationEval` evaluates a given derivation φ at a homogeneous differential form of degree 1, an element of $\Omega^1_{R/k}$, call it ω. Therefore, we first have to compute the following representation of ω:

$$\omega = \sum_{i=1}^{n} s_i dx_i,$$

where the s_i are elements in R. This can be done by using `difformCoef`.

Then we know that

$$\varphi(\omega) = \sum_{i=1}^{n} s_i \varphi(dx_i)$$

The images of the generators dx_i under φ are stored in a list `L`, which we assigned to φ since it is of type `derivation`. The procedure `derivationEval` computes the sum $\sum_{i=1}^{n}$ `s_i L[i]` which is exactly $\varphi(\omega)$.
There is also a shortcut for the evaluation. If `phi` is of type `derivation` and `df` of type `difform`, then we may write for the evaluation of `phi` at `df`:

```
phi(df);
```

C.5 The Lie derivative

In this section we present procedures that can compute the *Lie derivative* of a given derivation. In fact, we do this in two steps: first, we compute the *contraction* of the derivation, then we can easily compute the *Lie derivative*.

Definition C.5.1. Let R be as before and $\varphi : \Omega^1_{R/k} \to R$ a derivation.

a) Consider the map

$$\begin{aligned} (\Omega^1_{R/k})^t &\longrightarrow \bigwedge^{t-1} \Omega^1_{R/k} \\ (\omega_1, \ldots, \omega_t) &\longmapsto \sum_{i=1}^{t} (-1)^{i+1} \varphi(\omega_i) \cdot \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t, \end{aligned}$$

where $\widehat{\omega_i}$ denotes that ω_i is omitted from the wedge product. This map is alternating t-linear. Hence, by the universal property of the exterior power $\bigwedge^t \Omega^1_{R/k}$, we obtain an R-linear map:

$$\begin{aligned} i_\varphi^{(t)} : \bigwedge^{t} \Omega^1_{R/k} &\longrightarrow \bigwedge^{t-1} \Omega^1_{R/k} \\ \omega_1 \wedge \cdots \wedge \omega_t &\longmapsto \sum_{i=1}^{t} (-1)^{i+1} \varphi(\omega_i) \cdot \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_t \end{aligned}$$

When we combine all these maps, we obtain a graded R-linear map of degree -1, $i_\varphi : \Omega_{R/k} \to \Omega_{R/k}$.
The map i_φ is called **contraction of** φ. A graded part $i_\varphi^{(t)}$ is called **t-th contraction of** φ.

b) The **Lie derivative of** φ is then defined by:

$$L_\varphi = i_\varphi \circ d + d \circ i_\varphi,$$

where d denotes the universal derivation.

We reduce the computation of the Lie derivative to that of the contraction. Note that we computed the universal derivation before, using the procedure `difformDiff`.
Since the contraction i_φ is R-linear, we may focus on the application of i_φ to generators:

Remark C.5.2. Let $\varphi : \Omega^1_{R/k} \to R$ be a derivation and i_φ its contraction. If $\omega \in \Omega_{R/k}$, we can compute a representation:

$$\omega = \sum_{j=0}^{n} \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq n} s_{\nu_1 \ldots \nu_j} dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$$

using `difformCoef`. Hence, we obtain:

$$i_\varphi(\omega) = \sum_{j=0}^{n} \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq n} s_{\nu_1 \ldots \nu_j} i_\varphi^{(j)}(dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}) \tag{C.2}$$

and $i_\varphi^{(j)}(dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}) = \sum_{i=1}^{j} (-1)^{i+1} \varphi(dx_{\nu_i}) \cdot dx_{\nu_1} \wedge \cdots \wedge \widehat{dx_{\nu_i}} \wedge \cdots \wedge dx_{\nu_j}$

The remark shows how a procedure that applies $i_\varphi^{(j)}$ to a generator of degree j should look like. Exactly those computations are done by the procedure `difformContractionGen`: it takes a derivation φ and a generator $g = dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j}$. Then it checks, which degree-1 generators dx_i are part of g and computes $\varphi(dx_i)$ by `derivationEval`. The products $dx_{\nu_1} \wedge \cdots \wedge \widehat{dx_{\nu_i}} \wedge \cdots \wedge dx_{\nu_j}$ are computed by `difformDivision`. We simply divide the differential form $dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_i} \wedge \cdots \wedge dx_{\nu_j}$ by dx_{ν_i}.

The procedure `difformContraction` takes a derivation φ and a differential form ω and computes a representation of ω as in Remark C.5.2. Then it computes $i_\varphi^{(j)}(dx_{\nu_1} \wedge \cdots \wedge dx_{\nu_j})$ by using `difformContractionGen`. It multiplies by the coefficients $s_{\nu_1 \ldots \nu_j}$ and sums everything up. We obtain a representation as in (C.2) and hence, the procedure actually computes $i_\varphi(\omega)$.

Computing the Lie derivative is then an application of the procedures `difformContraction` and `difformDiff`: the procedure `difformLieDer` takes a derivation `phi` and a differential form `df` and computes:

```
difformContraction(phi,difformDiff(df)) +
 difformDiff(difformContraction(phi,df))
```

This corresponds to the definition of the Lie derivative and therefore, the procedure computes $L_{\texttt{phi}}(\texttt{df})$. It can be called via the shortcut `diff`.

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