Exercises to the lecture
Complexity Theory
Sheet 9

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Dr. Prakash Saivasan Delivery until 17.01.2017 at 10h

Exercise 9.1 (Unbounded Fan-In)

Let g be a gate in a circuit. The **Fan-In** of g is the in-degree of g, the number of incoming edges. A circuit has Fan-In **bounded by** $k \in \mathbb{N}$ if for any gate in the circuit, the Fan-In is bounded by k. In the lecture we considered circuits with Fan-In bounded by 2. This exercise shows that we can always restrict to this case:

Let C be a circuit with n input variables and unbounded Fan-In. Moreover, let $size(C) = s$ and $depth(C) = d$. Show that there is a circuit C' that has Fan-In bounded by 2 and

- $C'(x) = C(x)$ for all inputs x,
- $size(C') \in \mathcal{O}(s^2)$ and
- $depth(C') \in \mathcal{O}(d \cdot log\, s)$.

In particular, if $s(n)$ is a polynomial and $d(n)$ is a constant, we get: $depth(C') \in \mathcal{O}(log\, n)$.
Hint: Gates of Fan-In greater than 2 must be replaced. How can you do this ? You also need a bound for the maximal Fan-In of a gate in C.

Exercise 9.2 (Addition with parallel carry computation)

In this exercise we want to solve the **addition problem** using circuits:
Input: 2n variables $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, the binary representation of two natural numbers a and b.
Output: $n + 1$ variables $s_1, \ldots, s_{n+1}$, the binary representation of $s = a + b$.

A first approach to this problem would use *full adders*. A full adder for the i-th bits would compute $a_i + b_i + c_i$, where c_i is the carry, and it would output the sum bit and a new carry bit. This new carry bit could then be used as input for the full adder for the $(i + 1)$-st bits. This circuit would have depth $\mathcal{O}(n)$. We want to do better:

a) Construct a circuit $\mathcal{G}_i$ with unbounded Fan-In that computes the i-th carry bit c_i and has size $\mathcal{O}(i)$ and constant depth.
Hint: In contrast to the circuit described above, the computation of c_i should not depend on c_{i-1}. Note that c_i is 1 if and only if there is a position $j < i$, where the carry is generated and propagated to position i. Construct a Boolean formula for this condition - this may also depend on $a_1, \ldots, a_{i-1}$ and $b_1, \ldots, b_{i-1}$. Then transform the formula into a circuit.

b) Use Part a) to construct a circuit for the addition problem that has size $\mathcal{O}(n^2)$ and constant depth.

c) Conclude that there is a circuit of Fan-In bounded by 2 that solves the addition problem and has polynomial size and logarithmic depth.

Exercise 9.3 (Logspace reductions and the class NC)

Let A, B be two languages so that $A \leq_m^{log} B$ and $B \in \mathsf{NC}$. Show that in this case, also A is in NC.

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