Complexity Theory

Two goals:

↳ Study computational models and programming constructs
in order to understand
their power and limitations

↳ Study computational problems wrt.
their inherent complexity

Usually, complexity means time and space requirements (on a particular model).

Can also mean other measures like
randomness, number of alternations, circuit size.

Background:

↳ Theory of computability goes back to
Presburger, Church, Kleene, Post, Gödel, Turing, 1st half 20th century.

↳ Complexity theory goes back to
Hartmanis, Stearns:
"On the computational complexity of algorithms"
(use multitape TM, argue that concepts apply
to any reasonable model of computation).

1. Lower Bounds via Crossing Sequences

Goal: Prove an unconditional lower bound
on the power of a (uniform) complexity class.

Unconditional: • Showing an NP-hardness result means
"the problem is hard, assuming $P \neq NP$".
• Our result does not need such a condition.

Uniform: • One algorithm (TM) for all inputs.
• Non-uniform models may use different algorithms (circuits)
for different instances.

Note: Do not know other unconditional lower bounds,
even proving $SAT \notin DTIME(n^3)$
seems out of reach.

Approach: Employ a counting technique
called crossing sequences
for proving lower bounds
(vaguely related to fooling sets in automata theory).

1.1 Crossing Sequences

Let $COPY := \{w\#w \mid w \in \{a,b\}^*\}$.

The language is not context free (and hence not regular)

Goal: ↳ Upper bound: COPY can be decided in quadratic time
↳ Lower bound: COPY cannot be decided in subquadratic time
(on a deterministic 1-tape TM).

Recall:
(1) When we refer to a problem written as a set,

deciding the problem means deciding membership for that set.

Given $x \in \{a,b\}^*$, does $x \in COPY$ hold?

(b) The time / space is measured relative to the size of the input. Without further mentioning, the size will be referred to as n.

(c) O - notation = asymptotic upper bound ($\leq$) "no more than"

$$O(g(n)) := \{ f : \mathbb{N} \to \mathbb{N} \mid \exists c \in \mathbb{N} \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : f(n) \leq c \cdot g(n) \}$$

o - notation = asymptotic strict upper bound ($<$) "less than"

$$o(g(n)) := \{ f : \mathbb{N} \to \mathbb{N} \mid \underline{\forall} c \in \mathbb{R}^+ \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : f(n) \leq c \cdot g(n) \}.$$

Claim: $COPY \in DTIME(n^2)$

We assume that all tapes are right-infinite. To the left, they are marked by $\$$ (from there only move right, do not change)

Let M be a $t(n)$-time-bounded 1-tape TM for COPY ($t : \mathbb{N} \to \mathbb{N}$). We assume that M halts on the $\$$ marker (left).

Definition: A crossing sequence of M on input x at position i is the sequence of states of M when moving its head from cell i to cell $i+1$ or from cell $i+1$ to cell i. Denote the sequence by $CS(x, i)$.

Note: If q is a state in an odd position of the crossing sequence, then M moves its head from left to right.
If it is in an even position, M moves from right to left.

Lemma:
Let $x = x_1.x_2$ and $y = y_1.y_2$.
If $CS(x, |x_1|) = CS(y, |y_1|)$,
then $$x_1.x_2 \in L(M) \quad \text{iff} \quad x_1.y_2 \in L(M).$$

Proof:
Intuitively, the crossing sequence is all that M remembers about x_2 resp. y_2 when operating on the left part x_1.
Since the crossing sequences are assumed to coincide, M will behave the same on x_1, independent of whether x_2 or y_2 is on the right.
Since the $\#$ symbol is on the left, we are done. $\square$

Theorem: $COPY \notin DTIME(o(n^2))$.

Proof:
Let M be a deterministic 1-tape TM for COPY.
Consider inputs of the form
$$x = w_1.w_2 \# w_1.w_2 \quad \text{with } |w_1| = |w_2| = n.$$
For all $v \neq w_2$ with $|v| = |w_2|$, we have
$$CS(x, i) \neq CS(w_1.v \# w_1.v, i) \text{ for all } 2n+1 \leq i \leq 3n.$$
Otherwise, M would accept
$$w_1.w_2 \# w_1.v \quad \text{for some } v \neq w_2 \text{ with } |v| = |w_2|$$
by previous lemma.

We have

$$\boxed{\mathrm{Time}_M(x) \geq \sum_{i \geq 0} |CS(x,i)|}$$ (equality holds if head is always moved)

With this,

$$\sum_{w_2 \in \{a,b\}^n} \mathrm{Time}_M(w_1 w_2 \# w_1 w_2)$$

$$\geq \sum_{w_2} \sum_{\nu=2n+1}^{3n} |CS(w_1 w_2 \# w_1 w_2, \nu)|$$

$$= \sum_{\nu=2n+1}^{3n} \sum_{w_2} |CS(w_1 w_2 \# w_1 w_2, \nu)|.$$

- Fix some ν with $2n+1 \leq \nu \leq 3n$.
  For all w_2, the crossing sequences
  $$CS(w_1 w_2 \# w_1 w_2, \nu)$$
  are pairwise distinct (see above).
  Let l_ν be the average length of such a sequence, which means
  $$\sum_{w_2} |CS(w_1 w_2 \# w_1 w_2, \nu)| = 2^n \cdot l_\nu.$$
- If l_ν is the average length of a crossing sequence, at least half of the crossing sequences have length $\leq 2 l_\nu$ (see below).

There are
$$\leq (|Q|+1)^{2 \cdot l_\nu}$$
($+1$ if state is absent).
crossing sequences of length $\leq 2 l_\nu$

Hence, $$\boxed{(|Q|+1)^{2 l_\nu} \geq \frac{2^n}{2}}$$ // at least half of the (different) crossing sequences // have length $\leq 2 l_\nu$.

Thus, $l_\nu \geq c \cdot n$, for an appropriate $c > 0$, dependent on $|Q|$ (not on n and ν) (to enforce the inequality, we need that the crossing sequences are different)

- This yields

$$\sum_{w_2} \mathrm{Time}_M(w_1 w_2 \# w_1 w_2) \geq \sum_{v=2n+1}^{3n} \sum_{w_2} |CS(w_1 w_2 \# w_1 w_2, v)|$$

$$= \sum_{v=2n+1}^{3n} 2^n l_v$$

$$\geq \sum_{v=2n+1}^{3n} 2^n \cdot c \cdot n$$

$$= 2^n \cdot c \cdot n^2.$$

- Since there are only 2^n words w_2,

we get $\mathrm{Time}_M(w_1 w_2 \# w_1 w_2) \geq c \cdot n^2$

for at least one w_2. □

<u>Lemma:</u>

If $\frac{\sum_{i=1}^{n} w_i}{n} = d$, then at least half of the w_i

have a value $\leq 2d$.

<u>Proof:</u>

Assume at least half of the w_i have a value $> 2d$.

Then $\sum_{i=1}^{n} w_i > \frac{n}{\cancel{2}} \cdot \cancel{2}d = nd$.

From this $\frac{\sum_{i=1}^{n} w_i}{n} > \frac{\cancel{n}d}{\cancel{n}} = d$ ↯ to the assumption $\frac{\sum_{i=1}^{n} w_i}{n} = d$

□

1.2 A Gap Theorem for Deterministic Space Complexity

Goal: Show that $o(\log\log n)$ work tape is no better than no tape at all.

Theorem: $DSPACE(o(\log\log n)) = DSPACE(O(1))$.

Inclusion $\supseteq$ can be solved as homework.

We show $\subseteq$.

Note: The input tape is read-only.

Definition:
A small configuration of a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$
consists of
- ↳ the current state (in Q)
- ↳ the content of the work tape (in Γ^*)
- ↳ and the head position of the work tape.

We neglect
- ↳ the input and
- ↳ the head on the input tape.

Definition:
The extended crossing sequence of M on input x at position i
is the sequence of small configurations of M
when moving its head from cell i to $i+1$
or from cell $i+1$ to i on the input tape.

Denote this sequence by $ECS(x, i)$.

Proof (of the theorem):
Towards a contradiction, assume there is a Turing machine M
with space bound $s(n) \in o(\log\log(n)) \setminus O(1)$.

This means $s : \mathbb{N} \to \mathbb{N}$ is unbounded ($\forall m \in \mathbb{N}\ \exists n \in \mathbb{N} : s(n) > m$).

The number of small configurations on inputs of length n is

$$\leq |Q|\, |\Gamma|^{s(n)} \cdot (s(n)+2)$$

We have

$$|Q|\, |\Gamma|^{s(n)} \cdot (s(n)+2) \leq c^{s(n)} \quad \text{for large enough } n,$$

where c is a constant depending only on $|Q|$ and $|\Gamma|$.
This holds as $s(n)$ is unbounded.

- In an extended crossing sequence, no small configuration may appear twice in the same direction.
  Otherwise, a (large) configuration of M would appear twice in the computation, and as M is deterministic it would be stuck in an infinite loop (if M is assumed to halt)
  Thus, there are at most

$$\underbrace{(c^{s(n)}+1)^{c^{s(n)}}}_{\text{left}} \cdot \underbrace{(c^{s(n)}+1)^{c^{s(n)}}}_{\text{right}}$$

$$= (c^{s(n)}+1)^{2 \cdot c^{s(n)}}$$

$$\leq 2^{2^{d\, s(n)}}$$

different extended crossing sequences on inputs of length n, where $d > 0$ is a constant.

- For large enough n_0,

$$s(n) < \frac{1}{2d} \log \log n \quad \text{for all } n \geq n_0.$$

Hence,

$$2^{2^{d\,s(n)}} < 2^{2^{d\cdot\frac{1}{2d}\log\log n}}$$
$$= \left(2^{2^{\log\log n}}\right)^{\frac{1}{2}}$$
$$= n^{\frac{1}{2}}$$
$$\leq \frac{n}{2}$$

- Choose s_0 so that

$$s_0 > \max\{s(n) \mid 0 \leq n \leq n_0\}$$

and so that there is an input x with

$\mathrm{space}_M(x) = s_0$. < Such an s_0 exists because $s(n)$ is unbounded.

Let x be the shortest input with

$$s_0 = \mathrm{Space}_M(x).$$

- By the definition of s_0, we have

$$|x| > n_0,$$

for otherwise $\mathrm{Space}_M(x) = s_0 > s(|x|) \geq \mathrm{space}_M(x)$. ↯ (space bound)

Hence, by the definition of n_0 the number of crossing sequences is $< \frac{|x|}{2}$.

This means there are distinct positions $i < j < k$ with

$$ECS(x,i) = ECS(x,j) = ECS(x,k).$$

Indeed, if there were at most two positions for each extended crossing sequence, the length of x would be bounded by $|x| < \cancel{2}\cdot\frac{|x|}{\cancel{2}} = |x|$. ↯

- We shorten the input by glueing the crossing sequences either at i and j or at j and k.

On at least one of the two new inputs,
M will use the same space.
Why? Every small configuration on x
appears in at least one of the shortened strings.
↯ x was assumed to be the shortest string
with that space usage. $\square$

<u>Claim</u>:

$L := \{ bin(0) \# bin(1) \# \dots \# bin(n) \mid n \in \mathbb{N} \}$

is

↳ not regular

↳ in $DSPACE(\log \log (n))$.

The above theorem can be phrased as:

"If M runs in $o(\log \log (n))$ space,
then M accepts a regular language".

This relationship is non-trivial and relies on the following.

<u>Theorem</u>: $DSPACE(O(1)) = REG$.

The inclusion from right to left ($\supseteq$) is immediate.
For the reverse inclusion, the challenge is to mimic the behavior of a TM by reading every letter only once
(the TM may visit the letter several times).

Proof

$\Leftarrow$ $REG \subseteq DSPACE(O(1))$ is trivial.

The idea is to just ignore the constant space

$\Rightarrow$. $DSPACE(O(1)) \subseteq REG$.

Assumptions

(1) We assume that our TM has no extra tape. Indeed the constant space can be eliminated by including it as part of the state space

(2) We also assume that the TM is halting and that it always halts on the last symbol on the input tape.

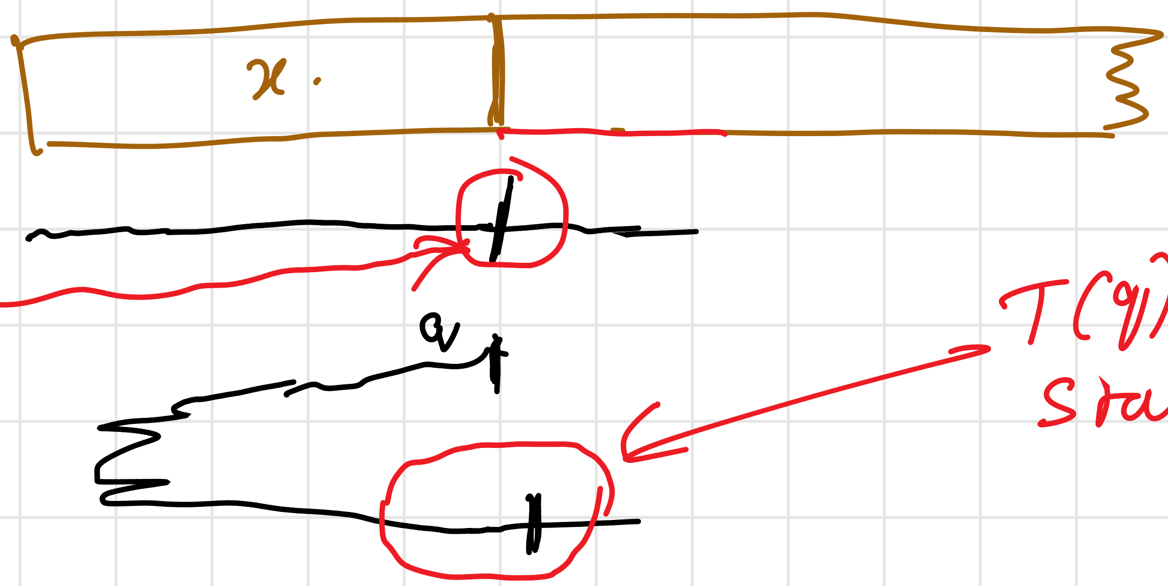
Defn: (crossing function): For any $x \in \Sigma^*$, we define the crossing function

$T_x : Q \cup \{\bullet\} \to Q$ as follows.

$T(\bullet)$ = Is the state that TM leaves x for the first time.

$T(q)$ = Is the state that TM leaves x for the first time after it enters x in state q.

$T(\bullet)$ is the state here.



$T(q)$ is the state here

Note that $T(\bullet)$ & $T(Q)$ can only depend on x & nothing else.

Now given a word x, it has a unique crossing function.

The set of all possible crossing func. is upper bounded by $(|Q|+1)^{|Q|}$ & hence is finite.

Also note that if $T_x = T_y$ then $\forall z$, $xz \in L(M)$ iff $yz \in L(M)$.

Now we define an equivalence relation over Σ^* as $\forall x, y \in \Sigma^*$, $x \equiv y$ iff $T_x = T_y$.

Such an equivalence relation satisfies the properties needed by Myhill-Nerode Thm.

From this we have $L(M)$ is Regular.