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Exercises to the lecture
Complexity Theory
Sheet 4

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Delivery until 20.11.2017 at 18h

Exercise 4.1 (Alternation Bounded QBF II)

Recall the definition of *alternation bounded* QBF of Exercise 3.2. There, we have shown that Σ_iQBF (Π_iQBF) lies in Σ_i^{P} (Π_i^{P}). In this exercise, you shall prove that Σ_i (Π_iQBF) is Σ_i^{P}-hard (Π_i^{P}-hard) with respect to polynomial time reductions.

Hint: Take an arbitrary language in Σ_i^{P} and reduce it to Σ_iQBF. In the lecture we have shown that QBF is PSPACE-complete. Extract the idea from this proof.

Exercise 4.2 (Collapse of the Polynomial Hierarchy)

In this exercise, we prove what is sufficient for a collapse of the polynomial hierarchy.

a) Assume we have an $i \in \mathbb{N}$ so that $\Sigma_i^{\mathsf{P}} = \Sigma_{i+1}^{\mathsf{P}}$. Show that $\Pi_i^{\mathsf{P}} = \Pi_{i+1}^{\mathsf{P}}$.

b) Prove the following: If we have an $i \in \mathbb{N}$ with $\Sigma_i^{\mathsf{P}} = \Pi_i^{\mathsf{P}}$, then for any $i' \geq i$ we have that $\Sigma_{i'}^{\mathsf{P}} = \Pi_{i'}^{\mathsf{P}} = \Sigma_i^{\mathsf{P}}$. Hence, the polynomial hierarchy collapses to the i-th level.
Hint: Prove the statement by an induction on i'. For the induction step, make use of the fact that $\Sigma_{i'+1}$QBF is $\Sigma_{i'+1}^{\mathsf{P}}$-hard.

c) Show that the existence of an $i \in \mathbb{N}$ with $\Sigma_i^{\mathsf{P}} = \Sigma_{i+1}^{\mathsf{P}}$ is already sufficient to cause a collapse of the polynomial hierarchy to the i-th level.

Exercise 4.3 (Minimal Boolean formulas)
Two Boolean formulas are called *equivalent* if they have the same value under any assignment to the variables. A formula φ is called *minimal* if there is no smaller formula that is equivalent to φ. We set $\mathsf{EQUIV} = \{(\varphi, \psi) \mid \varphi \text{ and } \psi \text{ are equivalent}\}$ and $\mathsf{MIN} = \{\varphi \mid \varphi \text{ is minimal}\}$.

a) Show that EQUIV is in Π_1^{P}.

b) Conclude that MIN is a problem in Π_2^{P}.

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