WS 2015/2016 16.12.2015

Exercises to the lecture
Complexity Theory
Sheet 8

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Christmas Exercise

Exercise 8.1 (Alternation bounded *QBF* and collapsing of the polynomial hierarchy)

Consider the following definition:

- $\Sigma_i QBF = \{\psi \mid \psi = \exists \overline{x_1} \forall \overline{x_2} \ldots Q_i \overline{x_i} \varphi(\overline{x_1}, \ldots, \overline{x_i}) \text{ is true } \}$,
- $\Pi_i QBF = \{\psi \mid \psi = \forall \overline{x_1} \exists \overline{x_2} \ldots Q_i \overline{x_i} \varphi(\overline{x_1}, \ldots, \overline{x_i}) \text{ is true } \}$,

where $\overline{x_j}$ denotes a finite sequence of variables and Q_i is a quantor. Note that there are at most $i-1$ alternations of quantors.
These *alternation bounded QBF* problems will help us to understand the polynomial hierarchy in more detail:

a) Show that $\Sigma_i QBF$ is in Σ_i^{P} and that $\Pi_i QBF$ is in Π_i^{P}.

b) Prove that $\Sigma_i QBF$ is Σ_i^{P}-hard with respect to polytime reductions and that $\Pi_i QBF$ is Π_i^{P}-hard with respect to polytime reductions.
Hint: Take an arbitrary language in Σ_i^{P} and reduce it to $\Sigma_i QBF$. Note that we showed that QBF is PSPACE*-complete. Extract the idea from this proof.*

Note that we now have the following situation:

- $SAT = \Sigma_1 QBF$ and this is $\mathsf{NP} = \Sigma_1^{\mathsf{P}}$-complete,
- co-$SAT = \Pi_1 QBF$ and this is co-$\mathsf{NP} = \Pi_1^{\mathsf{P}}$-complete.
- The alternation bounded *QBF* instances are complete for Σ_i^{P} and Π_i^{P},
- and the general *QBF* which allows unbounded alternation is PSPACE-complete.

We can make use of this to show that in some situations, the polynomial hierarchy collapses:

c) Assume we have a k so that $\Sigma_k^{\mathsf{P}} = \Sigma_{k+1}^{\mathsf{P}}$, then we also have $\Pi_k^{\mathsf{P}} = \Pi_{k+1}^{\mathsf{P}}$.

d) If we have a k so that $\Sigma_k^{\mathsf{P}} = \Pi_k^{\mathsf{P}}$, then we have for any $k' \geq k$ that $\Sigma_{k'}^{\mathsf{P}} = \Pi_{k'}^{\mathsf{P}} = \Sigma_k^{\mathsf{P}}$. So the polynomial hierarchy collapses to this level.
Hint: Prove this by induction on k'. Show that $\Sigma_{k+1} QBF$ is already in Σ_k^{P}.

e) If we have a k so that $\Sigma_k^{\mathsf{P}} = \Sigma_{k+1}^{\mathsf{P}}$ then we already have that the polynomial hierarchy collapses to this level.
Hint: Use parts d) and c).

We wish you a merry Christmas and a good start to the new year. Enjoy your vacation!

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