Exercises to the lecture
Complexity Theory
Sheet 9

Prof. Dr. Roland Meyer
M.Sc. Peter Chini Delivery until 13.01.2016 at 12h

Exercise 9.1 (co-Oracles)
Let $\mathcal{C}$ be a complexity class. Show that using oracles for $\mathcal{C}$ is equivalent to using oracles for co-$\mathcal{C}$:

a) Prove that $\mathsf{NP}^B = \mathsf{NP}^{\bar{B}}$ for any problem B in $\mathcal{C}$.

b) Conclude that we have: $\mathsf{NP}^{\mathcal{C}} = \mathsf{NP}^{\text{co-}\mathcal{C}}$.

Exercise 9.2 (Minimal Boolean formulas)
Two Boolean formulas are called **equivalent** if they have the same value on any assignment to the variables. A formula φ is called **minimal** if there is no smaller formula that is equivalent to φ.
Consider the problem:

$$MIN = \{\varphi \mid \varphi \text{ is minimal}\}.$$

a) Show that deciding whether two formulas are equivalent is in co-NP.

b) Prove that the co-problem $NOTMIN = \{\varphi \mid \varphi \text{ is not minimal}\}$ is in $\mathsf{NP}^{\mathsf{NP}}$.
Hint: Use Exercise 1.

c) Conclude that MIN is a problem in Π_2^{P}.

Exercise 9.3 (NP-intermediate languages)

Consider again the definition of the function $H : \mathbb{N} \to \mathbb{N}$, where

$$H(n) = \begin{cases} \text{minimal } i < loglog\, n \text{ so that for any input } x \in \{0,1\}^* : |x| \leq log\, n \\ \text{we have that } M_i \text{ computes } SAT_H(x) \text{ in } i \cdot |x|^i \text{ steps,} \\ \text{or } loglog\, n \text{ if no such } i \text{ exists.} \end{cases}$$

a) Show that $(log\, n)^{loglog\, n} \leq n$.
Hint: You may need that $log\, n \leq \sqrt{n}$. You can use this fact without any proof.

b) Prove that the function H is computable in polynomial time.

c) Recall the problem SAT_h from the lecture, where h is a polynomial-time computable function such that $\lim_{n\to\infty} h(n) = \infty$. Show the following: if SAT_h is NP-complete then SAT is in P.
Hint: This exercise is hard and therefore voluntary. For those who want to do it: Note that there is a polynomial-time reduction from SAT to SAT_h. A SAT-instance φ is mapped to a SAT_h-instance $\psi 01^{m^{h(m)}}$, where m is the size of ψ. Make use of the fact that the size of the SAT_h instance is at most the time that the reduction takes but keep in mind that $h(m)$ is not bounded. What does this mean for the size of ψ compared to the size of φ ? Note that you may use the reduction again to compress ψ even more.

Delivery until 13.01.2016 at 12h into the box next to 34-401.4