24. The Nemhauser – Trotter Theorem

Goal: · Algorithms are faster if they run on smaller kernels.
· Reduce the kernel for VERTEXCOVER.

Idea: Instead of local reduction rules
use rules on the structure of the whole graph.

Theorem (Nemhauser & Trotter '75):

Let $G=(V,E)$ have n vertices and m edges.
In time $O(\sqrt{n}\cdot m)$ we can compute disjoint sets
of vertices X and H so that

(1) If $D \subseteq H$ is a vertex cover of $G|_H$ (induced subgraph)
then $D \uplus X$ is a vertex cover of G.

(2) There is a minimum-sized vertex cover of G
containing X.

(3) The minimum-sized vertex cover for $G|_H$
has size $\geq \frac{|H|}{2}$.

Corollary (Chen, Kanj, Jia '01):

· VERTEXCOVER has a kernel of size $2k$.
· We can compute it in quadratic time.

Corollary (What Nemhauser & Trotter really says?):

· Every minimal vertex cover of G that contains X
takes the shape

$$X \uplus D,$$

where D is a minimal vertex cover of $G|_H$.
· Moreover, such a minimal vertex cover (containing X) exists.

<u>Proof:</u>

- Take a minimal vertex cover C of G that contains X.
  Exists by (2), this is the moreover part.
- Then the remaining graph $G \setminus X$ is covered by the set $C \setminus X$.
  Since $X \cap H = \emptyset$, we have
  $$(G \setminus X)|_H = G|_H .$$
  This means $(C \setminus X) \cap H$ is a vertex cover of $G|_H$.
- We now show that $C \setminus X$ is a minimal vertex cover of $G|_H$ and hence coincides with $(C \setminus X) \cap H$.
  Towards a contradiction, assume there is $D \subseteq H$ that is a vertex cover of $G|_H$ and satisfies $|D| < |C \setminus X|$.
  Then
  $D \uplus X$ is a vertex cover of G by (1).
  Moreover,
  $$\begin{aligned} |D \uplus X| &= |D| + |X| \\ &< |C \setminus X| + |X| = |C|. \ \lightning \text{ minimality of } C. \end{aligned}$$

$\square$

<u>Proof (of the Chen, Kanj, Jia corollary):</u>

The kernel is the subgraph $G|_H$ induced by H.
If it has size $> 2k$, by Nemhauser & Trotter (3) the answer is "no".

$\square$

For the proof of Nemhauser & Trotter, we need some graph theory.

Definition:

- A matching in a bipartite graph is a set of edges such that no two edges are incident to a common vertex.
- A matching is maximum-sized if there is no larger matching.

Theorem (Micali & Vazirani '80):

A maximum matching can be found in time $O(\sqrt{|V|} \cdot |E|)$.

We are interested in the relationship between matching and vertex cover.

Theorem (König's minimax theorem):

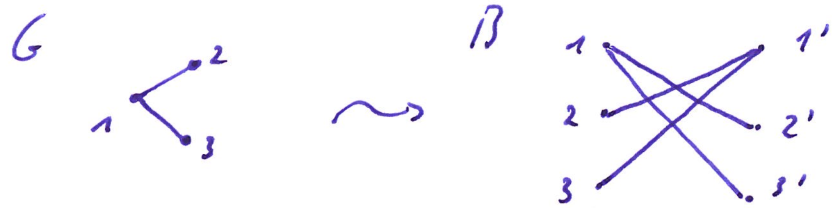
In a bipartite graph,
the size of a minimum-sized vertex cover
= the size of a maximum-sized matching.

Note: Minimax theorems always go like this:

- You know that things of type A always have size less than things of type B.
- Now assume you find a thing of type A with size equal to a thing of type B.
- Then the type A thing must be maximal-sized and the type B thing must be minimal-sized.

Algorithm (for Nemhauser & Trotter):

- For $G=(V,E)$, construct a bipartite graph $B=(V,V',E_B)$
  with $V' := \{v' \mid v \in V\}$ and $E_B := \{\{x,y'\},\{x',y\} \mid \{x,y\} \in E\}$.



- Compute a maximum-sized matching for B
  using Micali & Vazirani.
- By König's minimax theorem,
  this gives an optimal vertex cover C_B of B.
- Define

$$X := \{x \in V \mid x \in C_B \wedge x' \in C_B\}$$

$$H := \{x \in V \mid x \in C_B \vee x' \in C_B\} \setminus X.$$

$\square$

We omit here

↳ the proof of Micali & Vazirani

↳ the proof of König's minimax theorem (also conversion needed)

↳ the proof of correctness (Properties (1)–(3))
for the choice of X and H.