3.3 Perfectness and Reachability

Goal: · The intermediate accepting $\mathbb{N}$-run yields a solution to the characteristic equations.

· We want to show that also the reverse holds, provided the MGTS satisfies a condition called perfectness.

Idea: · The characteristic equations cannot guarantee non-negativity.

· Achieve non-negativity by adding pumping sequences.

· Perfectness thus has to guarantee

↳ the existence of pumping sequences and

↳ that the char. equations admit the pumping.

Definition:

An MGTS W is perfect, if

for every precovering graph G in W:

(i) $CSup(G) \neq \emptyset \neq CSdown(G)$

(ii) $supp(Char(W))$ justifies the unboundedness in G.

The latter means

$$\forall c \in C. \quad G.c_{\substack{in\\out}}(c) = \omega \quad \Rightarrow \quad x[G, \substack{in\\out}, c] \in supp(Char(W))$$

$$\forall t \in G.T. \qquad x[t] \in supp(Char(W)).$$

Theorem (Lambert's iteration lemma):

Let W be perfect.

Then $IAcc_{\mathbb{N}}(W) \neq \emptyset$ iff $IAcc_{\mathbb{Z}}(W) \neq \emptyset$

iff $Char(W)$ is feasible over $\mathbb{Z}$.

Proof:

We show how to turn
a solution to the characteristic equations
into an intermediate accepting $\mathbb{N}$-run.

- For simplicity, we assume the MGTS
is a single precovering graph G
and the initial and final configurations
are concrete, $c_{in}, c_{out} \in \mathbb{N}^C$.

- Let s_c be a solution to the characteristic equations
that exists by feasibility.
Let s_h be a full support solution
that exists by a lemma we had earlier.

- We define
$$s = s_c + s_h.$$
Now s also solves the characteristic equations.
Moreover, it satisfies $s[t] > 0$ f.a. $t \in T$.
With the Euler-Kirchhoff result,
we can turn s into a path ρ
with $\Psi(\rho) = s$.

- The path ρ may not be an $\mathbb{N}$-run,
because ω-decorated counters may fall below zero.

- By perfectness, however, there is $u \in \mathrm{CSup}(G)$
that has a positive effect on all ω-decorated counters.
The idea is to repeat u.

- We cannot repeat u in isolation. Otherwise we end up with a run that no longer solves reachability.
- We select $n \in \mathbb{N}$ so that

$$n \cdot sh[T] - \psi(u) - \psi(v) > 0 \quad (1)$$
$$\uparrow$$
$$CSdown(G)$$

This means n-copies of the support solution contain enough edges to fit in: u, v, and another cycle w.

- The idea is now to embed ρ into a repetition

$$u^k . \rho . w^k . v^k .$$

Unfortunately, we do not even know that $u.w.v$ is an $\mathbb{N}$-run. The word w may have a negative effect on the ω-decorated counters.

- <u>But</u>: By the definition of $CSdown(G)$, we know that v has a strictly negative effect on the ω-decorated counts. Since the overall effect of $u.w.v$ is zero, we know that uw must have a strictly positive effect on the ω-decorated counters.

- We select l so that

$$\mathrm{eff}(u^l.w^l) = \mathrm{eff}((u.w)^l)$$

is large enough to enable ω.

- We then add a pair u - w around $u^l.w^l$, and obtain $u.u^l.w^l.w$ with

$$\mathrm{eff}(u.u^l.w^l.w) > \mathrm{eff}(u^l.w^l).$$

We can thus repeat the addition k-times,

$$u^k.u^l.w^l.w^k,$$

and have the guarantee that the suffix w^k is enabled.

- The prefix u^k is always enabled. Moreover, since $u \in (\mathrm{Sup}(G))$, it has a strictly positive effect on the ω-decorated counters. This shows that we can select k so as to enable $u^l.w^l$.
- We can even make k large enough so that $u^k.\varrho.w^k.v^k$ is an $\mathbb{N}$-run.

<u>Case: $C_{out}[c] = \omega \neq C_{in}[c]$</u>

- In this case, the precovering graph should provide arbitrarily large values for c.
- By perfectness, $x[G, out, c] \in \mathrm{supp}(\mathrm{Char}(G))$.
- Then $u.w.v$ will produce a strictly positive value for this counter.

Case MGTS:

- When moving from precovering graphs to MGTS, we have to deal with ω-entries in the initial valuations. The problem is that the upcovering sequence $u \in CSup(G)$ may have a negative effect on the corresponding counter.
- To be able to execute u, we let the precovering graphs that are placed earlier than G in the MGTS produce a high-enough value. This works because the counter is in the support.
- Formally, we scale s_h by a factor m that not only achieves (1) from above, but also

$$m \cdot s_h[G, in, c] + eff(u) \geq 1$$
$$\wedge\ m \cdot s_h[G, out, c] - eff(v) \geq 1, \quad \text{f.a. } c \in \Omega(G).$$

As a consequence of this argumentation, there is $h_0 \in \mathbb{N}$ so that for all $h \geq h_0$:

$$(W.cinl). u_0^h . g_0 . w_0^h . v_0^h . up_1 \ldots up_{last} . u_{last}^h . g_{last} . w_{last}^h . v_{last}^h \in T_{R_{C \in \mathbb{N}}}(W).$$

□