Concurrency Theory (WS 2016)	Out: Sat, 26 Nov Due: Wed, 30 Nov

Exercise Sheet 5

D'Osualdo, Lederer, Schneider   Technische Universität Kaiserslautern

Problem 1: Minimal elements

Let $(Q, \leq)$ be a qo. For $A \subseteq Q$, we say $x \in A$ is minimal in A if there is no $x' \in A$ with $x' < x$. The upward closure of $A \subseteq Q$ is the set $A\uparrow := \{x \in Q \mid x \geq x' \text{ for some } x' \in A\}$.

Consider the following statements:

① Every strictly decreasing sequence over Q is finite and every antichain is finite.

② For every $A \subseteq Q$ there is a finite set $B \subseteq A$ of elements minimal in A such that $A \subseteq B\uparrow$.

③ $(Q, \leq)$ is wqo.

From the lecture we know that ③ $\implies$ ①, so prove the three statements equivalent by proving ① $\implies$ ② and ② $\implies$ ③.

Problem 2: Words are wqo

Let $(Q, \leq)$ be a wqo. For $u = u_1 \cdots u_m, v = v_1 \cdots v_n \in Q^*$, we write $u \leq^* v$ if there are $1 \leq i_1 < \cdots < i_m \leq n$ with $u_j \leq v_{i_j}$ for all $j = 1, \ldots, m$.

Derive that $(Q^*, \leq^*)$ is a wqo as a corollary of the lemmas proved in the lecture.

Problem 3: Multisets are wqo

A (finite) multiset over X is a function $m\colon X \to \mathbb{N}$ such that the set $\lfloor m \rfloor := \{x \in X \mid m(x) > 0\}$ is finite. We denote by $\mathcal{M}(X)$ the set of such multisets. Let $(X, \leq_X)$ be a quasi order and $m_1, m_2 \in \mathcal{M}(X)$, an *embedding* from m_1 to m_2 is an injective function $\phi\colon \lfloor m_1 \rfloor \to \lfloor m_2 \rfloor$ such that $x \leq_X \phi(x)$ and $m_1(x) \leq m_2(\phi(x))$ for all $x \in \lfloor m_1 \rfloor$. We define $m_1 \leq_{\mathcal{M}(X)} m_2$ to hold when there exists an embedding from m_1 to m_2.

Prove that $(\mathcal{M}(X), \leq_{\mathcal{M}(X)})$ is a wqo if $(X, \leq_X)$ is a wqo.

[Hint: *adapt the proof seen in the lecture for finite sets*]

[Bonus: *there is a shorter proof* ☺]