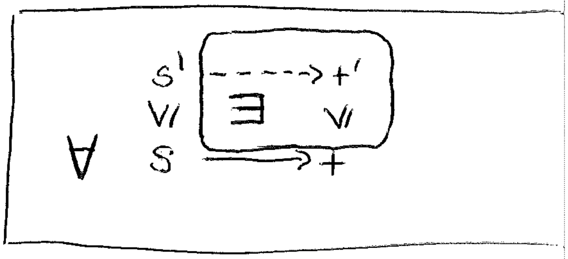
Well Structured Transition Systems (WSTS)

Def A Quasi Ordered Transition System (QOTS) $(S, \rightarrow, \leq)$ has the following properties:

① S is a set of configurations (typically infinite).
② $\rightarrow \subseteq S \times S$ is a transition relation.
③ $\leq \subseteq S \times S$ is a qo.

Def A relation $\leq$ over S is a simulation if $\forall s, t, s'$ such that $s \rightarrow t$ and $s' \geq s$ we have $\exists t' \in S$ $s' \rightsquigarrow t'$ and $t' \geq t$.
We call $\leq$ a
① (weak) simulation [DEFAULT] if $\rightsquigarrow = \rightarrow^*$
② transitive simulation if $\rightsquigarrow = \rightarrow^+$
③ strong simulation if $\rightsquigarrow = \rightarrow$

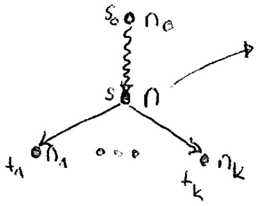


Def We call a QOTS $(S, \rightarrow, \leq)$ a Well Structured Transition System (WSTS) if
① $\leq$ is a simulation
② $(S, \leq)$ is a wqo
Moreover, we call a WSTS weak [DEFAULT], transitive or strong according to $\leq$.
Moreover, we call a WSTS effective if $\rightarrow, \leq$ are decidable

TERMINATION FOR WSTS

Def Let $s_0 \in S$. The finite reach tree $FRT(s_0)$ form s_0 is a S-labelled directed tree so that

ⓐ root n_0 is labelled by s_0

ⓑ 

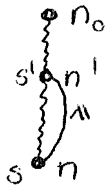
→ n has a child for each transition $s \longrightarrow t_i$

if $\nexists n'$ in the path from n_0 to n that is labelled with s' s.th $s' \leq s$

n has no child

otherwise

⇝ we say n is subsumed by n'



Remark: $s, s', t_1, \dots, t_k \in S$ are labels and $n', n, n_0, \dots, n_k$ are the vertices of the FRT

Lemma $FRT(s)$ is finite for every fin tely branching WSTS.

Proof by König's Lemma: If the $FRT(s)$ is infinite, it contains an infinite path. By wqo $(S, \leq)$ such an infinite path would contain a subsumed vertex. But that vertex has no successor by construction ↯

Lemma A transitive WSTS $(S, \longrightarrow, \leq)$ has a non terminating computation tree from $s_0 \in S$ if and only if $FRT(s_0)$ contains a subsumed node.

Proof (⇒) Let π be an infinite computation from s_0. Then $\pi = \pi_1 \pi_0$ where π_1 is finite and labels a maximum path from the root of $FRT(s_0)$. Since the last node in this path is a leaf, and has no successors (by existence of π), n is subsumed.

(⇐) By transitive simulation we have

$$\underset{n_0}{s_0} \longrightarrow^* \underset{n'}{t_1} \longrightarrow^+ \underset{n}{t_2} \geq t_1 \Rightarrow s_0 \longrightarrow^* t_1 \longrightarrow^+ t_2 \longrightarrow^+ t_3 \geq t_2$$

In other words

$$\begin{array}{ccc} t_2 & \longrightarrow^+ & t_3 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ t_1 & \longrightarrow^+ & t_2 \end{array}$$

gives rise to an infinite computation.

Theorem: TERMINATION from $s_0 \in S$ is decidable for a transitive effective WSTS $(S, \rightarrow, \leq)$.

Remark: We need effective to construct the $FRT(s_0)$.