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Games with perfect information
Exercise Sheet 13 – Solution

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Exercise 1: Gale-Stewart games as graph games

Let $\mathcal{G}(A, B)$ be a Gale-Stewart game. Define an equivalent game over a graph with the set of positions

a) $V = A \times \{\square, \bigcirc\}$,

b) $V = A^*$.

In each case, specify the ownership, the arcs, the winning condition, and the initial position of interest.

Proof:

a) We give a straightforward encoding of the Gale-Stewart game.

- Vertices $V = A \times \{\square, \bigcirc\}$.
- Ownership: $V_\square = A \times \{\square\}$, $V_\bigcirc = A \times \{\bigcirc\}$.

  The second component of a position $(a, \star)$ indicates the active player.
- Arcs: $R = \{ ((a, \star), (b, \overline{\star})) \mid a, b \in A, \{\square, \bigcirc\} = \{\star, \overline{\star}\} \}$.

  The active player can select an arbitrary action $a' \in A$ without any restriction. The players alternately take turns.
- The problem is that we need an initial position in V. This position contains some action, but actually we want refuter to select the first action.

  Let us select some $a_0 \in A$ and fix $(a_0, \bigcirc) \in V_\bigcirc$ as the initial position of interest. The second component indicates that refuter starts. The first component will be ignored for the winning condition.
- A maximal play is an infinite sequence $p \in V^\omega$. We define a function

$$\begin{aligned} \mathrm{proj}_A \quad : \quad V &\to A \\ (a, \star) &\mapsto a \end{aligned}$$

  that projects away the second component. We extend it to sequences in the natural way, i.e. by applying it to every entry of the sequence. We obtain the function

$$\mathrm{proj}_A \colon V^\omega \to A^\omega .$$

  This allows us to associate to a play $p \in V^\omega$ the sequence $\mathrm{proj}_A(p) \in A^\omega$. We need to get rid of the dummy position a_0 that is at the beginning of each play. To this end, we define another function

$$\begin{array}{rrcl} \text{drop} & : \; A^\omega & \longrightarrow & A^\omega \\ & a.s & \longmapsto & s \end{array}$$

that removes the first entry from an infinite sequence. We can now define the winning condition by

$$\begin{array}{rrcl} win & : \; V^\omega & \longrightarrow & \{\square, \bigcirc\} \\ & p & \longmapsto & \begin{cases} \bigcirc, & \text{drop}\big(\text{proj}_A(p)\big) \in B\,, \\ \square, & \text{else, i.e. } \text{drop}\big(\text{proj}_A(p)\big) \notin B\,. \end{cases} \end{array}$$

b) The use A^* to keep track of finite prefixes of the plays. The difficulty is encoding the winning conditions

- Vertices $V = A^*$.
- Ownership:

$$\begin{array}{rrcl} \text{owner} & : \quad V & \longrightarrow & \{\square, \bigcirc\} \\ & (a, \text{☆}) & \longmapsto & \begin{cases} \bigcirc, & |p| \text{ is even }, \\ \square, & \text{else, i.e. } |p| \text{ is odd }. \end{cases} \end{array}$$

  Prover should pick the even positions $(0, 2, 4, \ldots)$. Note that in move i, the prefix of the play has length i. (Here, it is important that we start to count from 0.)

- Arcs: $R = \big\{ \big(p, p.a\big) \;\big|\; p \in A^*, a \in A \big\}$.

  The active play can prolong the play by an arbitrary action.

- The initial position of interest is the empty play $\varepsilon \in V$.

  Note that $|\varepsilon| = 0$, so indeed refuter has the first move.

- A maximal play is an infinite sequence $p \in V^\omega$. Each entry p_i is a finite play in A^* of length i. Note that each p_i is a prefix of p_j for $j > i$,

$$p = (\varepsilon).(a).(ab).(abc)\ldots$$

  To define the winning condition we need to consider the infinite word that occurs as the limit of the p_i.

  The i^{th} action is picked in move $i + 1$. We define the infinite word $\lim p \in A^\omega$ by

$$(\lim p)_j = (p_{j+1})_j\,.$$

  This means we pick $\lim p$ at position j to be the j^{th} entry of the first p_i that is long enough so that $(p_i)_j$ is defined (and this i is $j + 1$).

  We can finally define

$$
\begin{aligned}
win \quad : \quad V^{\omega} \quad &\to \quad \{\square, \bigcirc\} \\
p \quad &\mapsto \quad \begin{cases} \bigcirc\,, & \lim p \in B\,, \\ \square\,, & \text{else, i.e. } \lim p \notin B\,. \end{cases}
\end{aligned}
$$

$\square$

Remark: Notation for (sets of) sequences

We recall the notations needed for the next two exercises.

Let V be a set. We denote by V^* the set of sequences over V of finite length, by V^{ω} the set of sequences over V^{ω} of infinite length.

Let $p', p'' \in V^*, p \in V^{\omega}$. Finite sequences p', p'' can be concatenated, resulting in the finite-length sequence $p'.p''$. A finite sequence p' can be concatenated with the infinite sequence p, resulting in the infinite sequence $p'.p$.

For sets of sequences, we define their concatenation element-wise.
Let $K', K'' \subseteq V^*$ and $H \subseteq V^{\omega}$. We define

$$
\begin{aligned}
K'.K'' &= \left\{p'.p'' \in V^* \;\middle|\; p' \in K', p'' \in K''\right\}, \\
K'.H &= \left\{p'.p \in V^{\omega} \;\middle|\; p' \in K', p \in H\right\}.
\end{aligned}
$$

We identify elements $x \in V$ with the sequence $x \in V^*$ of length one.

For a sequence $p' \in V^*$, we write p' to denote the singleton set $\{p'\} \subseteq V^*$.

Exercise 2: Reachability games as Gale-Stewart games

Let $\mathcal{G}$ be a reachablity game, specified as usual by a game arena $G = (V_{\square} \cup\!\!\!\cdot\; V_{\bigcirc}, R)$ and a winning set $V_{reach} \subseteq V$. For simplicity, let us assume that G is bipartite and the player take turns alternately. Furthermore, we fix the initial position $x_0 \in V_{\bigcirc}$.

Our goal is to create an equivalent Gale-Stewart game $\mathcal{G}(V, B)$, where B is of the shape

$$
B = (B_{\square} \cup B_{reach}) \setminus (B_{\bigcirc} \cup B_{\neg x_0})\,.
$$

a) We define

$$
B_{\bigcirc} = \bigcup_{\substack{x \in V_{\bigcirc},\\ y \in V_{\square},\\ (x,y) \notin R}} \left\{p \in V^{\omega} \;\middle|\; p \in V^{odd}.x.y.V^{\omega}\right\}.
$$

Here, $V^{odd} \subseteq V^*$ should denote the set of all finite sequences over V of odd length.

Argue that an infinite play $p \in V^\omega$ of $\mathcal{G}(V, B)$ is in $B_\bigcirc$ if and only if refuter makes a move that is illegal, i.e. not corresponding to an arc in the graph. This means that there is a prefix of the play of the shape $p' = p''.x$ in which refuter y such that $(x, y) \notin R$.

b) Define the set $B_{\neg x_0}$ of plays that are not starting in x_0.

c) Define the set $B_\square$ of all plays in which prover makes an illegal move.

d) Define the set B_{reach} of all plays in which at least one position in the set V_{reach} occurs.

Proof:

a) Assume $\tilde{p} \in V^\omega$ is an infinite play.

- If $\tilde{p} \in B_\bigcirc$, we have that there are $x \in V_\bigcirc, y \in V_\square$ with $(x, y) \notin R$ such that

  $$\tilde{p} \in \left\{ p \in V^\omega \;\middle|\; p \in V^{odd}.x.y.V^\omega \right\}.$$

  This means we can write $\tilde{p} = p'.x.y.p''$ where p' is a finite play of odd length and $p'' \in V^\omega$ is an infinite suffix.

  Note that since refuter starts and the players alternately take turns, refuter picks the even positions of $\tilde{p}$. This also means that she has to pick whenever the finite play is of even length. Since $p'.x$ is of even length, this means the move (x, y) was picked by refuter.

  As we have $(x, y) \notin R$, this is not a valid move.

- Assume $\tilde{p}$ contains a illegal move made by refuter. As mentioned above, refuter always moves after even prefixes. We may write $\tilde{p}$ as

  $$\tilde{p} = p'.x'.y'.p''$$

  where $p'.x'$ is the even-length prefix (and thus p' is of odd length), and $x'.y'$ is the illegal move (i.e. $(x', y') \notin R$).

  We get

  $$\tilde{p} \in \left\{ p \in V^\omega \;\middle|\; p \in V^{odd}.x'.y'.V^\omega \right\} \subseteq B_\bigcirc.$$

b) We define

$$B_{\neg x_0} = (V \setminus \{x_0\}).V^\omega = \{p \in V^\omega \mid p_0 \neq x_0\}$$

as the set of all plays whose first entry is not x_0 (and the rest is arbitrary).

c) Similar to $B_\bigcirc$ given in the exercise, we define

$$B_\square = \bigcup_{\substack{x \in V_\square, \\ y \in V_\bigcirc, \\ (x,y) \notin R}} \{p \in V^\omega \mid p \in V^{even}.x.y.V^\omega\}.$$

Here, it is important that prover moves after odd prefixes $p'.x$. Correctness can be proven as in Part a).

d) We define

$$B_{reach} = V^*.V_{reach}.V^\omega = \{p \in V^\omega \mid \exists i \in \mathbb{N}\colon p_i \in V_{reach}\}$$

as the set of all plays that consist of a finite prefix, a position in V_{reach} and an arbitrary infinite suffix.

□

Exercise 3: Open sets

Let A be a set. We call a set $B \subseteq A^\omega$ of infinite sequences over A **open** if it is of the shape

$$B = K.A^\omega$$

for some set $K \subseteq A^*$ of finite sequences over A. (This essentially means that a set B is open if the membership of a play $p \in A^\omega$ in B is determined by a finite prefix of B.)

a) Prove that the empty set $\varnothing \subseteq A^\omega$ and A^ω itself are open.

b) Prove that if B and B' are open, then also their union $B \cup B'$ is open.

c) Prove that if B and B' are open, then also their intersection $B \cap B'$ is open.

Proof:

a)
- We may see the empty set $\varnothing \subseteq A^*$ as a set of finite sequences. If we concatenate it with anything, we obtain the empty set:

  $$\varnothing.H = \varnothing\ .$$

  This allows us to write $\varnothing = \varnothing.A^\omega$.

- There are many representations of A^ω as open set. A few examples:

  $$A^\omega = \{\varepsilon\}.A^\omega = A.A^\omega = A^*.A^\omega\ .$$

  To argue for correctness e.g. of the first representation, note that prepending ε does not change an infinite sequence.

b) Assume that B, B' are open. This means there are $K, K' \subseteq V^*$ such that

$$B = K.A^\omega \qquad\qquad B' = K'.A^\omega\ .$$

We claim their union $B \cup B'$ is open, as it can be written as

$$B \cup B' = (K \cup K').A^{\omega} .$$

Assume $p = B \cup B'$. We may write it as $p = p'.p''$ with p' in K or in K', proving $p \in (K \cup K').A^{\omega}$. The other direction is similar.

c) Assume that B, B' are open. This means there are $K, K' \subseteq V^*$ such that

$$B = K.A^{\omega} \qquad\qquad B' = K'.A^{\omega} .$$

We claim their intersection $B \cap B'$ is open.

One might think that, similar to Part b), one can show $B \cap B' = (K \cap K').A^{\omega}$. We get that one inclusion, namely $B \cap B' \supseteq (K \cap K').A^{\omega}$, holds, but the other one does not. If $p \in B \cap B'$, we get decompositions $p = p_{fin}.p_{inf}$ and $p'_{fin}.p''_{inf}$. We even know that p_{fin} is a prefix of p'_{fin} or the other way around, but since they may differ in length, they are not necessarily equal.

We need to consider finite sequences that are in one of the sets and have a prefix that is in the other set. We define

$$\begin{aligned} K_{\cap} = K_{\cap}^{(1)} \cup K_{\cap}^{(1)} = \quad & \left\{p_{fin} \in K \,\middle|\, p_{fin} = p_1.p_2 \text{ with } p_1 \in K', p_2 \in V^*\right\} \\ \cup \quad & \left\{p_{fin} \in K' \,\middle|\, p_{fin} = p_1.p_2 \text{ with } p_1 \in K, p_2 \in V^*\right\} , \end{aligned}$$

i.e. we take all sequences in K that can be prolonged to a sequence in K' and the other way around.

We can now formally prove

$$B \cap B' = K_{\cap}.A^{\omega} .$$

- Assume $\tilde{p} \in K_{\cap}.A^{\omega}$. We decompose $\tilde{p} = \tilde{p}_{fin}.\tilde{p}_{inf}$ with $\tilde{p}_{fin} \in K_{\cap}$.

  We consider the case $\tilde{p}_{fin} \in K_{\cap}^{(1)}$, the other case is analogous.

  Since $K_{\cap}^{(1)} = \left\{p_{fin} \in K \,\middle|\, p_{fin} = p_1.p_2 \text{ with } p_1 \in K', p_2 \in V^*\right\}$ is a subset of K, we obtain $\tilde{p}_{fin} \in K, \tilde{p} \in B$.

  It remains to prove $\tilde{p} \in B'$. We may write $\tilde{p}_{fin} = \tilde{p}_1.\tilde{p}_2$, where $\tilde{p}_1 \in K'$. This gives us a new decompositions $\tilde{p} = \tilde{p}_1.(\tilde{p}_2.\tilde{p}_{inf})$ with $\tilde{p}_1 \in K', \tilde{p}_2.\tilde{p}_{fin} \in A^{\omega}$, proving $\tilde{p} \in B'$.

- Assume $\tilde{p} \in (B \cap B')$. We get that there are decompositions $\tilde{p} = \tilde{p}_{fin}.\tilde{p}_{inf}$ and $\tilde{p} = \tilde{p}'_{fin}.\tilde{p}''_{inf}$ with $\tilde{p}_{fin} \in K, \tilde{p}'_{fin} \in K'$.

  If $\tilde{p}_{fin}$ and $\tilde{p}'_{fin}$ are of the same length, they have to be equal, and we have $\tilde{p}_{fin} \in K \cap K'$. Since $K \cap K'$ is a subset of $K_{\cap}$ (we chose p_2 to be ε in the definition of $K_{\cap}^{(1)}$), this proves $\tilde{p} \in K_{\cap}.A^{\omega}$.

  Otherwise, one of the two finite prefixes is shorter, let us say wlog. $\tilde{p}_{fin}$. (The other case is analogous.) This allows us to write $\tilde{p}'_{fin} = \tilde{p}_{fin}.\tilde{p}_2$ for some $\tilde{p}_2$v, as $\tilde{p}_{fin}$ is a prefix of $\tilde{p}'_{fin}$.

Since $\tilde{p}'_{fin} \in K'$ and $\tilde{p}_{fin} \in K$, this proves $\tilde{p}_{fin'} \in K_{\cap}^{(2)} \subseteq K_{\cap}$.

We conclude $\tilde{p} = \tilde{p}'_{fin}.\tilde{p}'_{inf} \in K_{\cap}.A^{\omega}$.

$\square$

Remark

The Parts a) - c) of Exercise 3 almost prove that the notion of being open as defined here defines a topology on A^{ω}.

It remains to prove that arbitrary unions of open sets are open. This is also true, and the proof is a straightforward extension of the proof given for Part b). Consider an arbitrary union of open sets, i.e.

$$B = \bigcup_{i \in \mathcal{I}} B_i \,,$$

where $\mathcal{I}$ is some index set. For each $i \in \mathcal{I}$, we know that we can write $B_i = K_i.A^{\omega}$ for some $K_i \subseteq A^*$.

We obtain

$$B = \bigcup_{i \in \mathcal{I}} B_i = \left(\bigcup_{i \in \mathcal{I}} K_i \right).A^{\omega} \,.$$

Note that the proof for Part b) cannot be extended to show that open sets are closed under arbitrary (or even just countable) intersections: We could define a set $K_{\cap}$ similar as above, but the proof of $\bigcap_{i \in \mathcal{I}} B_i = K_{\cap}.A^{\omega}$ will fail. In our proof, we have considered the longer one of the prefixes p_{fin}, p'_{fin}. If we have infinitely many of these prefixes, their length may grow unboundedly. Consequently, it is not possible to pick the "longest" one.

Remark

The sets $B_{\bigcirc}, B_{\neg x_0}, B_{\square}, B_{reach}$ from the Exercise 2 are open.

For $B_{\neg x_0}$ and B_{reach}, this is already clear by the way we presented the sets.

For $B_{\bigcirc}$ and $B_{\square}$, we need the remark above which shows that unions of open sets are again open.