Out: June 6 Due: June 13

Submit your solutions on Wednesday, June 13, at the beginning of the lecture.
Please submit in groups of three persons.

Exercise 1: Language emptiness as a parity game

Consider the PTA A_2 from Part b) of Exercise 9.42 in the lecture notes (resp. Exercise 1 from Exercise sheet 7). Recall that $A_2 = (\Sigma, \{q_+, q_-\}, q_+, \rightarrow, \Omega)$ where

$$\begin{aligned} \rightarrow_a &= \{(q_+, (q_-, q_-))\} , \\ \rightarrow_b &= \{(q_+, (q_+, q_-)),\ (q_+, (q_-, q_+)),\ (q_-, (q_-, q_-))\} , \\ \Omega(q_-) = 0 \,, &\qquad \Omega(q_+) = 1 \,. \end{aligned}$$

To avoid deadlocks, transform the automaton to a language-equivalent automaton A' that has at least one transition $(q, \vec{q}) \in \rightarrow_a$ for each source state q and symbol a.

Recall that we can construct a parity game $\mathcal{G}(A')$ that is won by $\bigcirc$ if and only if the language of A' is non-empty (see Proposition 9.30 in the lecture notes).

Construct the parity game $\mathcal{G}(A')$ and identify a positional winning strategy for the existential player. How does the tree described by the strategy look like?

Exercise 2: Making finite-memory strategies uniform

Let $\mathcal{G}$ be a game on some finite graph $G = (V_\square \uplus V_\bigcirc, R)$ with some arbitrary fixed winning condition *win*. Assume for each of the two positions $x, y \in V$, player $\bigcirc$ has some finite-memory strategy, say induced by the transducers T_x and T_y, respectively.

Show how to construct a transducer T such that the finite-memory strategy induced by T is winning from both x and y.

Exercise 3: Constructing a transducer

Consider the game $\mathcal{G}(n)$ (for some $n \in \mathbb{N}, n > 0$) on the following graph:

$$
\begin{aligned}
V &= \{guess, go\} \uplus N \uplus X \quad \text{with}\\
N &= \{1, \ldots, n\}\,,\\
X &= \{x_1 \ldots x_n\}\,,\\
R &= \{(guess, i), (i, go) \mid i \in \{1, \ldots, n\}\} \cup \{(go, x_i) \mid i \in \{1, \ldots, n\}\} \cup \big\{(x_i, x_j) \mid i, j \in \{1, \ldots, n\}\big\}\,,
\end{aligned}
$$

$$owner(guess) = owner(i) = \square \text{ for all } i \in N\,,$$

$$owner(go) = owner(x_i) = \bigcirc \text{ for all } x_i \in X\,.$$

Let us focus on plays starting in position *guess*. Note that all maximal plays from this position are infinite, and they visit exactly one position from the set N, and they visit this position exactly once.

Such a play p is won by $\bigcirc$ if and only $|\mathrm{Inf}(p) \cap X| = m$ holds, where m is the unique position from N that occurs in p.

a) Draw G for $n = 4$. Assume that the universal player picks the move $(guess, 3)$. Draw in a positional strategy for $\bigcirc$ that wins under this assumption.

b) Let $n \in \mathbb{N}, n > 0$ be an arbitrary fixed number, and consider $\mathcal{G}(n)$. Show how to construct a transducer T such that the finite-memory strategy for $\bigcirc$ induced by T is winning from *guess*.

Example 4: An expensive game

Let $n \in \mathbb{N}, n > 0$ be a fixed positive number. We define a Muller game $\mathcal{G}^{\text{Muller}}$ on the game arena $G = (V, R)$ with $V = \{1, \ldots, n\} \times \{\bigcirc, \square\}$ (where the second component indicates the active player) and the moves defined by

$$R = \Big\{(i, \star) \to (j, \overline{\star}) \;\Big|\; i, j \in \{1, \ldots, n\}, \{\bigcirc, \square\} = \{\star, \overline{\star}\}\Big\}\,.$$

The Muller judgment is defined as follows: $judgment(X) = \bigcirc$ if and only if

$$|X \cap V_{\bigcirc}| = \max\{i \mid (i, \square) \in X\}\,.$$

a) Draw G for $n = 2$.

b) Explain the winning condition in your own words.

c) Construct the parity game $\mathcal{G}^{\text{parity}}$ obtained form $\mathcal{G}^{\text{Muller}}$ by the LAR construction. You can fix some initial LAR lar_0 and just draw all LARs reachable from lar_0. Similar to the example in the lecture, mark all positions with their priorities.

   Draw in a positional winning strategy for the existential player $\bigcirc$.