SS 2012 18. April 2012

Exercises to the lecture Logics
Sheet 1

Jun.-Prof. Dr. Roland Meyer Due 2. Mai 2012 12:00 Uhr

Exercise 1.1 [Structural Induction]

The *depth* $t(A)$ of a formula A is defined as follows.

- If A is atomic, then $t(A) = 0$.
- If $A \equiv (B * C)$ for a binary connective $*$, then

$$t(A) = \max\{t(B), t(C)\} + 1.$$

- If $A \equiv \neg(B)$, then $t(A) = t(B) + 1$.

Furthermore, let $|A|$ be the length of the formula A, i.e., the number of symbols in A, counting parentheses.

Prove by structural induction that in every correctly bracketed formula

a) the number of opening and the number of closing parentheses coincide.

b) $|A| \leqslant 5k + 1$, where k is the number of occurrences of connectives in A.

c) $|A| \leqslant 4 \cdot 2^{t(A)} - 3$.

Exercise 1.2 [Semantics of formulae]

a) Let φ be a valuation with $\varphi(p) = 1$ and $\varphi(q) = \varphi(r) = 0$. Calculate

$$\varphi((r \vee (\neg(p \wedge q))))$$

step-by-step using the definition of the evaluation of valuations.

b) Prove or disprove that $p \rightarrow (q \rightarrow (p \vee r))$ is a tautology.

c) Prove or disprove $p \rightarrow q \vDash q \rightarrow p$.

d) Prove or disprove $p \vee q \mathrel{\vDash\!\!\!\Dashv} \neg(\neg p \wedge \neg q)$.

Exercise 1.3 [Deduction theorem]

a) Let $A_1, \ldots, A_n, B$ be formulae in propositional logic. Show that $A_1 \wedge \cdots \wedge A_n \vDash B$ provided that $(A_1 \rightarrow (A_2 \rightarrow (\cdots (A_{n-1} \rightarrow A_n) \cdots) \vDash B$.

b) Let Σ be a set of formulae and B a formula in propositional logic. Show that $\Sigma \vDash B$ if and only if $\Sigma \cup \{\neg B\}$ is unsatisfiable.

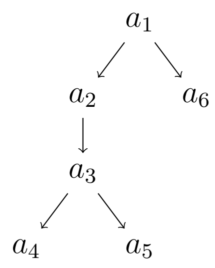
Exercise 1.4 [Paths in rooted trees]

A *rooted tree* is a tree in which one node is chosen as the *root* and the edges are directed such that their source is closer to the root than their target. A *rooted path* is a path that starts in the root. For each rooted path P, we write $\hat{P}$ for the set of nodes it meets. A subset of nodes is called *rooted path set* if it is of the form $\hat{P}$ for some rooted path P.

Let $V = \{a_1, \ldots, a_n\}$ be the nodes of a rooted tree and let $p_1, \ldots, p_n$ be atomic formulae. The subsets of V and the valuations on $p_1, \ldots, p_n$ are in one-to-one correspondence, where the set $S \subseteq V$ corresponds to the valuation φ for which

$$\varphi(p_i) = 1 \text{ if and only if } a_i \in S$$

for each $i \in \{1, \ldots, n\}$.



a) For the rooted tree on the right, present a formula A for which $\varphi(A) = 1$ if and only if φ corresponds to a rooted path set.

b) Devise a general method that, given a rooted tree T, constructs a formula A such that $\varphi(A) = 1$ if and only if φ corresponds to a rooted path set in T.

Delivery: until 2. Mai 2012 12:00 Uhr into the box next to room 34/401.4