In-class Exercises to the Lecture Logics
Sheet 5

Jun.-Prof. Dr. Roland Meyer Discussion on 21./22.6.2012

**Exercise 5.1** [MSO on words]

Let $\Sigma$ be an alphabet and $\Phi$ be the set of formulae in *second order* predicate logic in which only the following predicates occur:

- $p_a$, a unary predicate, for each $a \in \Sigma$,
- $<$, a binary predicate, and
- suc, a binary predicate.

To each word $w \in \Sigma^+$, we associate the interpretation $I^w = (D^w, I_c^w, I_v^w)$, where

- $D^w = \{1, \ldots, |w|\}$ is the set of positions in $w$,
- for each $a$ $I^w(p_a)$ is the set of positions that contain an $a$,
- $I^w(x < y) = 1$ if and only if $I^w(x) < I^w(y)$,
- $I^w(\mathrm{suc}(x, y)) = 1$ if and only if $I^w(y) = I^w(x) + 1$.

For example, let $\Sigma = \{a, b\}$ and $w = ab$. Then $D^w = \{1, 2\}$, $I^w(p_a) = \{1\}$ and $I^w(p_b) = \{2\}$. In this situation, we have

$$I^w \models \exists x \exists y (x < y \wedge p_a(x)) \wedge \neg \exists x \exists y \exists z (x < y \wedge y < z).$$

The *language defined by* $A \in \Phi$ is

$$L(A) = \{w \in \Sigma^+ \mid I^w \models A\}.$$

The logic defined this way is also known as *Monadic Second Order Logic* on words.

a) Present a formula $A$ with $L(A) = \{a\}^+\{b\}^+,\ \Sigma = \{a, b\}$.

b) Present a formula $A$ with $L(A) = \Sigma^*\{a\}\Sigma^*\{b\},\ \Sigma = \{a, b\}$.

c) Suppose you have symbols $r$, $a$, and $i$ in $\Sigma$, where $r$, $a$, and $i$ represent a *request*, an *acknowledge*, and an *internal* event, respectively. Present a formula $A$ such that $L(A)$ is the set of words such that: after each request event, at least one acknowledge event follows eventually.

**Exercise 5.2** [MSO on graphs]

a) Describe, similar to Exercise 5.1,
   - a collection of predicates,
   - an interpretation for each graph,

   such that you can define sets of graphs using formulae.

b) Present a formula that defines precisely the set of connected graphs.

**Exercise 5.3** [Undecidability]

A context-free grammar is called *linear* if in each rule, the right-hand side contains at most one occurrence of a nonterminal symbol. Show that the following problem is undecidable: Given linear context-free grammars $G_1$ and $G_2$, is $L(G_1) \cap L(G_2) = \varnothing$?

**Exercise 5.4** [Formulae in predicate logic]

a) Let $A \equiv \forall x \exists y p(x, y)$ and $B \equiv \exists y \forall x P(x, y)$. Which of these formulas is deducible from the other? Are they equivalent?
b) Is the formula $\forall x p(x) \to \exists x p(x)$ a tautology?