SS 2013 30. April 2013

Exercises to the lecture Logics
Sheet 2

Jun.-Prof. Dr. Roland Meyer Due 10. Mai 2013

Exercise 2.1 [Complete sets of connectives]

A and B are called *equivalent*, denoted $A \models\mid B$, if $\varphi(A) = \varphi(B)$ for any valuation φ. For a set C of connectives, let $\mathcal{F}(C)$ be the set of formulae that contain no connectives other than those in C. We call a set C of connectives *complete* if for any formula A, there is an equivalent formula $B \in \mathcal{F}(C)$.

a) Suppose you have the connective $\bar{\wedge}$ ("NAND"), which satisfies

$$\varphi(A \bar{\wedge} B) = 1 - \min\{\varphi(A), \varphi(B)\}$$

for any formulae A, B. Using structural induction, show that $\{\bar{\wedge}\}$ is a complete set of connectives.

b) For valuations φ_1, φ_2, we write $\varphi_1 \leqslant \varphi_2$ if $\varphi_1(p) \leqslant \varphi_2(p)$ for any propositional variable p. A formula A is said to be *monotone* if $\varphi_1 \leqslant \varphi_2$ implies $\varphi_1(A) \leqslant \varphi_2(A)$ for any valuations φ_1, φ_2 (in other words: the Boolean function corresdponding to A is monotone). Using structural induction, show that every formula in $\mathcal{F}(\{\wedge, \vee\})$ is monotone.

c) Deduce from b) that $\{\wedge, \vee\}$ is not a complete set of connectives.

d) Show that for any monotone formula A, there is an equivalent one in $\mathcal{F}(\{\wedge, \vee\})$ (Hint: Adapt the method for obtaining a DNF from a truth table and consider minimal satisfying valuations).

Exercise 2.2 [Logical equivalence]

Show that the logical equivalence is a congruence relation, that is: If $A \models\mid A'$ and $B \models\mid B'$, then also $\neg A \models\mid \neg A'$ and $(A * B) \models\mid (A' * B')$ for any binary connective $*$.

Exercise 2.3 [Horn formulae]

Suppose there are additional atomic formulae $\top$ and $\bot$, which satisfy

$$\varphi(\top) = 1 \quad \text{and} \quad \varphi(\bot) = 0$$

for every valuation φ. A *Horn formula* is a conjunction of formulae $(A \to B)$ such that A and B are each a propositional variable or one of the symbols $\top$ and $\bot$. An example of such a formula is

$$(p \to q) \wedge (\top \to p) \wedge (q \to p) \wedge (r \to \bot).$$

Devise a method that decides whether a given Horn formula is satisfiable and try to make it as time efficient as possible (Hint: Successively check off occurrences of propositional variables). Explain why your method is faster than trying valuations one by one.

Delivery: until 10. Mai 2013 into the box next to room 34/401.4