SS 2013 July 3, 2013

In-class Exercises to the Lecture Logics
Sheet 6

Jun.-Prof. Dr. Roland Meyer Discussion on July 4 and 5, 2013

Exercise 6.1 [The Compactness Theorem for predicate logic]

Let A be a formula in first order predicate logic such that for each $n \in \mathbb{N}$, A has a model $\mathcal{M}$ with $|\mathcal{M}| \geqslant n$.

a) Present for each $n \in \mathbb{N}$ a formula B_n such that for every structure $\mathcal{M}$, $\mathcal{M} \models B_n$ if and only if $|\mathcal{M}| \geqslant n$.

b) Consider the set $\Sigma = \{A \wedge B_n \mid n \in \mathbb{N}\}$. Using the compactness theorem for first order predicate logic, show that Σ is satisfiable.

c) Show that A has an infinite model.

d) Conclude that there is no formula F such that $\mathcal{M} \models F$ if and only if $\mathcal{M}$ has a finite domain.

Exercise 6.2 [A validity test]

Show that there is an algorithm that, given a formula of the form

$$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m B,$$

where B is quantifier-free and contains no function symbols of arity $\geqslant 1$, determines whether the formula is a tautology. *Hint:* Consider the Herbrand expansion.

Exercise 6.3 [Decidable theories]

Let Σ be a complete theory for which a recursively enumerable axiom system is available. Show that Σ is decidable. *Hint:* Consider derivations in the system $\mathcal{F}$.

Exercise 6.4 [Completeness and Consistency]

Show that a theory Σ is complete if and only if there is no formula A such that $T_{\Sigma \cup \{A\}}$ and $T_{\Sigma \cup \{\neg A\}}$ are consistent. *Note:* Hence, you have shown that completeness means that the theory cannot be extended consistently in two ways that contradict each other.