Recapitulation:

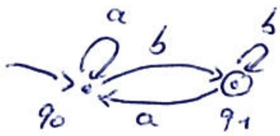
ω-regular languages

$$L = \underbrace{\bigcup_{i=0}^{n-1}}_{\text{finite union}} \underbrace{V_i}_{\text{regular prefix}} . W_i^{\omega} \leftarrow \omega\text{-iteration}$$

with $V_i, W_i \subseteq \Sigma^*$
and $W_i \cap \Sigma^+ \neq \emptyset$ f.a. $0 \leq i \leq n-1$

(Non) deterministic Büchi automata

$\mathcal{A}$

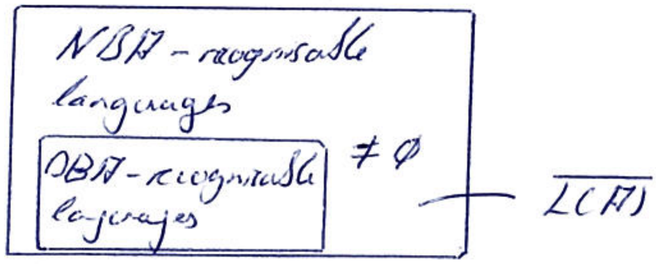


where $L(\mathcal{A})$ = words with infinitely many bs.

Runs $r = q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{b} q_1 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{a} \ldots$

accepting with $\mathrm{Inf}(r) \cap Q_F \neq \emptyset$

Realised



Theorem:

A language is recognised by an NBA iff it is ω-regular.

Proof: Homework

Direction $\Leftarrow$ requires some lemmas.

Lemma:

Let $\mathcal{A}, \mathcal{B}$ two NBAs. Then there is an NBA $\mathcal{A} \cup \mathcal{B}$ so that $L(\mathcal{A} \cup \mathcal{B}) = L(\mathcal{A}) \cup L(\mathcal{B})$.

Lemma:

Let $\mathcal{A}$ an NFA and $\mathcal{B}$ an NBA. Then there is an NBA $\mathcal{A}.\mathcal{B}$ with $L(\mathcal{A}.\mathcal{B}) = L(\mathcal{A}).L(\mathcal{B})$.

Lemma:

Let A an NPA with $L(A) \cap \Sigma^+ \neq \emptyset$.
Then there is an NBA A^ω with $L(A^\omega) = L(A)^\omega$.

Direction => of theorem by close examination of accepting runs.
Then

$$L(A) = \bigcup_{q \in Q_F} \dots \cdot \dots^\omega$$

By this theorem: remaining closure properties for ω-regular languages by automata constructions.

Closure under intersection:

Theorem:

Let A, B two NBAs with n and m states, respectively.
Then there is an NBA $A \| B$ with $3nm$ states
so that $L(A \| B) = L(A) \cap L(B)$.

Idea: Product automaton synchronises runs in A and B

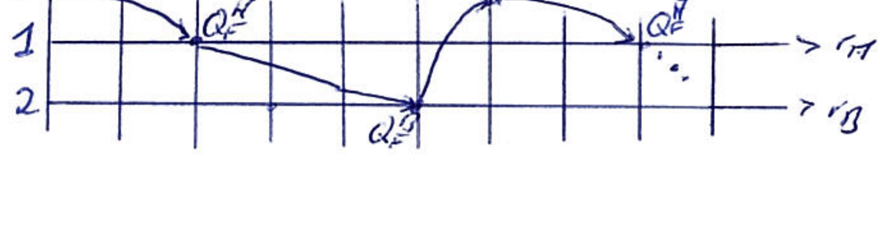
Problem: Accepting runs in A and in B can visit final states at different moments.

Solution: Add a counter to states that keeps track of
- whether no final state has been seen $(=0)$
- whether a final state of A has been seen $(=1)$
- whether a final state of A and B has been seen $(=2)$

(afterwards reset counter)

Accept whenever 2 has been seen.

Illustration:



Construction:

Let $\mathcal{A} = (Q_{\mathcal{A}}, q_0^{\mathcal{A}}, \to_{\mathcal{A}}, Q_F^{\mathcal{A}})$ and $\mathcal{B} = (Q_{\mathcal{B}}, q_0^{\mathcal{B}}, \to_{\mathcal{B}}, Q_F^{\mathcal{B}})$.

Define

$$\mathcal{A} \| \mathcal{B} := (Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \{0,1,2\}, (q_0^{\mathcal{A}}, q_0^{\mathcal{B}}, 0), \to, Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \{2\})$$

with

$$(q_{\mathcal{A}}, q_{\mathcal{B}}, i) \xrightarrow{a} (q_{\mathcal{A}}', q_{\mathcal{B}}', j) \text{ if } q_{\mathcal{A}} \xrightarrow{a}_{\mathcal{A}} q_{\mathcal{A}}' \text{ and } q_{\mathcal{B}} \xrightarrow{a}_{\mathcal{B}} q_{\mathcal{B}}' \text{ and}$$

$$j = \begin{cases} 1 & \text{if } i=0 \text{ and } q_{\mathcal{A}}' \in Q_F^{\mathcal{A}} \text{ or } i=1 \text{ and } q_{\mathcal{B}}' \notin Q_F^{\mathcal{B}}. \\ 2 & \text{if } i=1 \text{ and } q_{\mathcal{B}}' \in Q_F^{\mathcal{B}} \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Size $= |Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \{0,1,2\}| = |Q_{\mathcal{A}}|\,|Q_{\mathcal{B}}|\, 3 = 3nm$.

Language equality:

$L(\mathcal{A} \| \mathcal{B}) = L(\mathcal{A}) \cap L(\mathcal{B})$.

„$\supseteq$" Let $w = a_0 a_1 a_2 \ldots \in L(\mathcal{A}) \cap L(\mathcal{B})$.

Then there are accepting runs

$r_{\mathcal{A}} = q_0^{\mathcal{A}} \xrightarrow{a_0} q_1^{\mathcal{A}} \xrightarrow{a_1} q_2^{\mathcal{A}} \ldots$

$r_{\mathcal{B}} = q_0^{\mathcal{B}} \xrightarrow{a_0} q_1^{\mathcal{B}} \xrightarrow{a_1} q_2^{\mathcal{B}} \ldots$

They yield a run in $\mathcal{A} \| \mathcal{B}$:

$r = (q_0^{\mathcal{A}}, q_0^{\mathcal{B}}, i_0) \xrightarrow{a_0} (q_1^{\mathcal{A}}, q_1^{\mathcal{B}}, i_1) \xrightarrow{a_1} (q_2^{\mathcal{A}}, q_2^{\mathcal{B}}, i_2) \ldots$

Third component in states determined by $r_{\mathcal{A}}$ and $r_{\mathcal{B}}$.

Claim: r visits final states (with $i=2$) infinitely often.

Key insight:

- For every moment $j \in \mathbb{N}$ there is a later $k \in \mathbb{N}$ with $q_k^{\mathcal{A}} \in Q_F^{\mathcal{A}}$.
- ——————— " ——————— $q_k^{\mathcal{B}} \in Q_F^{\mathcal{B}}$.
- This holds as $r_{\mathcal{A}}$ and $r_{\mathcal{B}}$ visit final states infinitely often.
- With this argument, we find
  - from any state $(q_j^{\mathcal{A}}, q_j^{\mathcal{B}}, 0)$ a later state with $(q_k^{\mathcal{A}}, q_k^{\mathcal{B}}, 1)$ and
  - from $(q_k^{\mathcal{A}}, q_k^{\mathcal{B}}, 1)$ a later state $(q_\ell^{\mathcal{A}}, q_\ell^{\mathcal{B}}, 2)$.

"$\subseteq$" Decompose run

$$r = (q_0^A, q_0^B, 0) \xrightarrow{a_0} (q_1^A, q_1^B, i_1) \xrightarrow{a_1} \ldots$$

into two runs r_A, r_B of A and B.

By construction, counter evolves along $(0^+ . 1^+ . 2)^\omega$

↳ Infinitely many 1 means r_A accepting in A.

↳ Infinitely many 2 means r_B accepting in B. □

4.3 Complementation à la Büchi

Goal (ultimately): Inclusion checking $L(A) \subseteq L(B)$

$\subseteq L(C)$

- Recent algorithm by Fogarty and Vardi '10
- Relies on classical ideas of Büchi
  - ↳ Discuss Büchi's ideas first.
  - ↳ Solve complementation problem for Büchi automata (NBA).
  - ⇒ goal of this chapter.

Learned: Deterministic Büchi automata

↳ weaker than NBAs

↳ not closed under complementation

(cannot invert final states:

non-final states infinitely often

$\neq$

final states finitely many times)

Show: Non-deterministic Büchi automata closed under complementation:

Given NBA A, there is NBA $\bar{A}$ with $L(\bar{A}) = \overline{L(A)}$.

4.3.1 Some infinite combinatorics

Often required when dealing with termination

↳ WQO in concurrency theory

↳ check latest "Communications of the ACM".

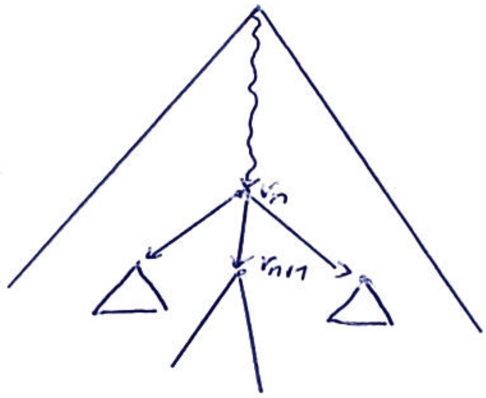
Theorem (König 1927)

Consider an infinite tree $T=(V,\to,r)$
of finite outdegree (i.e., for all $v\in V$ is $\{v' \mid v\to v'\}$ finite).
Then there is an infinite path in T.

Proof:

Construct path $v_0 . v_1 . v_2 \ldots$ inductively
so that for all $i\in\mathbb{N}$ we have
- v_{i+1} is child of v_i
- tree below v_i is infinite.

IA: $v_0 = r$ is root of an infinite tree.



IS: Assume we constructed $v_0 \ldots v_n$.
↳ The tree below v_n is infinite.
↳ But v_n has finite outdegree
$\Rightarrow$ There is a child v_{n+1} of v_n
that is root of an infinite tree
(otherwise ↯ to hypothesis and v_n
would not start an infinite tree).

□

Theorem (Ramsey 1930)

Let (V,E) complete infinite undirected graph.
Let C finite set of colors.
Let $f: E\to C$ some edge coloring.
Then there is an infinite complete subgraph $(V', \underbrace{E\cap \mathcal{P}(V')}_{E'})$
with $f(E')=c$ constant.

Proof:

Wlog. assume $V=\mathbb{N}$.
Fixed ordering $<$ on V, wlog. take $<$ on $\mathbb{N}$.
Let $C=\{1,\ldots,k\}$.

Construct infinite sequence

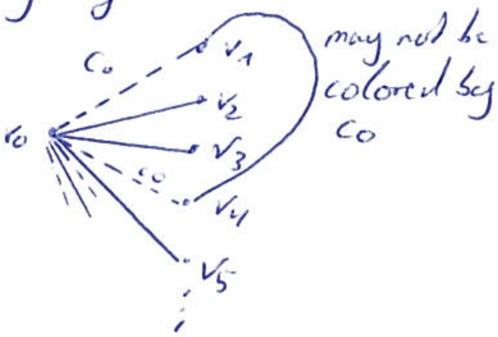
$(v_0, V_0, c_0), (v_1, V_1, c_1), \dots$

with

- $v_0 < v_1 < v_2 < \dots$
- $V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$ all V_i infinite
- $f(\{v_i, x\}) = c_i$ f.a. $x \in V_i$
- $v_{i+1} \in V_i$

Again proceed by induction

IA:
- Choose v_0 arbitrary
- c_0 = color that colors infinitely many edges connected to v_0
- $V_0 = \{x \in V \mid f(\{v_0, x\}) = c_0\}$



IS: Assume we already constructed

$(v_0, V_0, c_0), \dots, (v_n, V_n, c_n)$

Construct

$(v_{n+1}, V_{n+1}, c_{n+1})$

by choosing

- $v_{n+1} \in V_n$ with $v_n < v_{n+1}$ (works as V_n infinite)
- and reasoning as before with V_n as underlying infinite set.

In this infinite sequence, there is a color c that occurs infinitely often.

$\hookrightarrow$ Corresponding set $V' := \{v_i \mid (v_i, V_i, c_i) \text{ with } c_i = c\}$ is what we looked for.

Consider edge $\{v_i, v_j\}$ with $v_i < v_j$ both in V'.

$\hookrightarrow$ We have

$$v_j \in V_{j-1} \subseteq V_i$$

$\hookrightarrow$ Hence, $f(\{v_i, v_j\}) = c$.

□