Games with perfect information
Exercise sheet 6

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Out: May 22 Due: May 29

Submit your solutions on Wednesday, May 29, during the lecture.

Exercise 1: Weak parity games

A **weak parity game** is given by a game arena $G = (V_\square \uplus V_\bigcirc, R)$ and a priority function Ω. Instead of considering the highest priority that *occurs infinitely often* to determine the winner of a play, we consider the highest priority that *occurs at all*.

Formally, for a set A and an infinite sequence $p \subseteq A^\omega$ over A, we define the **occurrence set**

$$\text{Occ}(p) = \{a \in A \mid \exists i \in \mathbb{N} \colon p_i = a\} .$$

The winner of the weak parity game given by G and Ω is determined by the **weak parity winning condition**:

$$\begin{aligned} win \ : \ Plays_{max} &\to \{\bigcirc, \square\} \\ p &\mapsto \begin{cases} \bigcirc, & \text{if } \max \text{Occ}(\Omega(p)) \text{ is even}, \\ \square, & \text{else, i.e. if } \max \text{Occ}(\Omega(p)) \text{ is odd}. \end{cases} \end{aligned}$$

a) Present an algorithm that, given a weak parity game on a finite, deadlock-free game arena, computes the winning regions of both players. Briefly argue that your algorithm is correct.

   Hint: Attractors!

b) Is the winning condition of weak parity games prefix-independent, i.e. does Lemma 6.5 from the lecture notes hold?

   Do uniform positional winning strategies exist?

Algorithm: Zielonka's recursive algorithm

Input: parity game $\mathcal{G}$ given by $G = (V_\square, V_\bigcirc, R)$ and Ω.

Output: winning regions $W_\square$ and $W_\bigcirc$.

Procedure `solve`($\mathcal{G}$)

$n \leftarrow \max_{x \in V} \Omega(x)$
if $n = 0$ **then**
  return $W_\bigcirc = V, W_\square = \varnothing$
else
  $N = \{x \in V \mid \Omega(x) = n\}$
  if n even **then**
    $\star \leftarrow \bigcirc, \overline{\star} \leftarrow \square$
  else
    $\star \leftarrow \square, \overline{\star} \leftarrow \bigcirc$
  end if
  $A \leftarrow \mathrm{Attr}_{\star}^{\mathcal{G}}(N)$
  $W'_\bigcirc, W'_\square \leftarrow solve(\mathcal{G}\restriction_{V \setminus A})$
  if $W'_\star = V \setminus A$ **then**
    return $W_\star \leftarrow V, W_{\overline{\star}} \leftarrow \varnothing$
  else
    $B \leftarrow \mathrm{Attr}_{\overline{\star}}^{\mathcal{G}}(W'_{\overline{\star}})$
    $W''_\square, W''_\bigcirc \leftarrow solve(\mathcal{G}\restriction_{V \setminus B})$
    return $W_\star = W''_\star, W_{\overline{\star}} = W''_{\overline{\star}} \cup B$
  end if
end if

Exercise 2: Algorithmics of parity games

Use the algorithm algorithm to solve the following game. x^i means that position x has priority i.