Addendum:

Definition:

Let $A = (\Sigma, Q, q_0, \rightarrow, Q_F)$ be an NFA.

We generalize the transition relation to words

$$\rightarrow \subseteq Q \times \Sigma^* \times Q.$$

The definition is by induction on the length of the word:

- For all $q \in Q$: $q \xrightarrow{\varepsilon} q$.
- For all q_1, q_2, q_3, for all $w \in \Sigma^*$, $a \in \Sigma$:

  if $q_1 \xrightarrow{w} q_2$ and $q_2 \xrightarrow{a} q_3$,

  then $q_1 \xrightarrow{wa} q_3$.

2. Closure Properties of Regular Languages

2.1 Determinisation and Complementation

see previous notes.

2.2 Homomorphisms and ε-Transitions.

Definition:

A homomorphism is a function $h: \Sigma^* \rightarrow \Gamma^*$
so that for all $x, y \in \Sigma^*$

$$h(xy) = h(x)h(y).$$

Proposition: Consider $h: \Sigma^* \rightarrow \Gamma^*$ a homomorphism.

(1) $h(\varepsilon) = \varepsilon$.

(2) Every function $f : \Sigma \to T^*$ extends uniquely to a homomorphism $h_f : \Sigma^* \to T^*$.

(3) Every homomorphism is uniquely determined by its values on Σ.
Let $f = h|_\Sigma : \Sigma \to T^*$.
Then
$$h = h_f.$$

<u>Consequence:</u>
We only have to define how a homomorphism acts on the letters from Σ to specify it completely.

<u>Proof:</u>
- To show (1), note that $h(\varepsilon) = h(\varepsilon\varepsilon)$.
  We thus have
  $$|h(\varepsilon)| = |h(\varepsilon\varepsilon)| = |h(\varepsilon)\cdot h(\varepsilon)| = |h(\varepsilon)| + |h(\varepsilon)|.$$
  Thus $|h(\varepsilon)| = 0$, which means $h(\varepsilon) = \varepsilon$.
- Since $h_f(\varepsilon) = \varepsilon$ by (1), we have for $w = a_1 \dots a_n \in \Sigma^*$
  $$h_f(w) = h_f(a_1) \dots h_f(a_n) = f(a_1) \dots f(a_n).$$
- We have $f(a) = h(a)$ for all $a \in \Sigma$.
  By (2), there is only one homomorphism that we can obtain with this valuation of the letters, which means $h_f = h$. $\square$

Theorem (Closure under h and h^{-1}):

Consider $h : \Sigma^* \to \Gamma^*$ a homomorphism.

Let $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Gamma^*$ be regular languages.

Then $h(L_1) := \{ h(w) \mid w \in L_1 \} \subseteq \Gamma^*$ and (1)

$h^{-1}(L_2) := \{ w \in \Sigma^* \mid h(w) \in L_2 \} \subseteq \Sigma^*$ (2)

are regular.

Proof:

(1) Assume $L_1 = L(A)$ with A an NFA over Σ.

We replace every transition labelled by x by the automaton for $h(x)$ (potentially inserting new states or contracting states).

The result is the NFA $h(A)$ over Γ.

<u>Claim:</u> $L(h(A)) = h(L(A))$.

(2) Assume $L_2 = L(B)$ with B an NFA over Γ.

We shortcut every transition sequence

$$p \xrightarrow{h(x)} q \quad \text{to} \quad p \xrightarrow{x} q.$$

The result is an automaton $h^{-1}(B)$ over Σ.

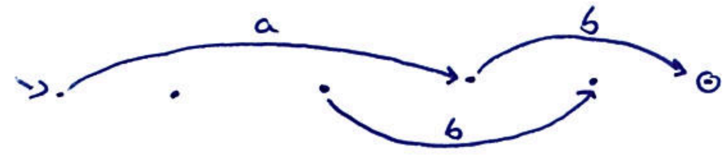
<u>Claim:</u> $L(h^{-1}(B)) = h^{-1}(L(B))$. $\square$

Example:

Let $\Sigma = \{a, b\}$ and $\Gamma = \{x, y, z\}$.

Let $h(a) = xyz$, $h(b) = zz$.

Consider B: 

Then $h^{-1}(B)$: 

Application:

We use homomorphisms to give a clean treatment of ε-transitions.

- Define an NFA with τ-transitions to be the structure
$$A = (\Sigma, \tau, Q, q_0, \rightarrow, Q_F),$$
where $\tau \notin \Sigma$ is a special symbol, and define
$$A_\tau := (\Sigma \cup \{\tau\}, Q, q_0, \rightarrow, Q_F)$$
to be the associated NFA over $\Sigma \cup \{\tau\}$.

- We say that A accepts $w \in \Sigma^*$, if
$\exists v \in (\Sigma \cup \{\tau\})^*$: • A_τ accepts v (using ordinary acceptance for NFAs).
  - w is obtained from v by erasing all τ letters.

    More formally:
    $$w = h(v)$$
    where $h: (\Sigma \cup \{\tau\})^* \rightarrow \Sigma^*$ is defined by
    $$h(\tau) := \varepsilon \quad \text{and} \quad h(a) := a \quad \text{for all } a \in \Sigma.$$

Lemma: $L(A) = h(L(A_\tau))$.

Note: One can also remove τ while doing a subset construction.